Mathematical Reviews

Vol. 17, No. 10 NOVEMBER 1956 pp. 1037-1168

HISTORY, BIOGRAPHY

Godeaux, L. La géométrie italienne en Belgique. Rend. Sem. Mat. Fis. Milano 25 (1953–54), 93–100 (1955).

Bouligand, Georges. Aspects courants de la recherche mathématique, indépendants de son objet. C. R. Acad. Sci. Paris 242 (1956), 2689–2692.

Holton, Gerald. Johannes Kepler's universe: its physics and metaphysics. Amer. J. Phys. 24 (1956), 340–351.

Bordoni, Piero Giorgio. Galileo instauratore della meccanica. Ricerca Sci. 26 (1956), 1378–1391.

Nicotra, Salvatore. Lorenzo Mascheroni (1750-1800). Sci. Giovani 5 (1955–56), 97–102.

Čebotarev, N. G. Mathematical autobiography. Acad. R. P. Romîne. An. Romîno-Soviet. Ser. Mat.-Fiz. (3) 9 (1955), no. 3, 84–116. (Romanian)

Translation of Uspehi Mat. Nauk 3(1948), no. 4(26), 3–66.

★**Wiener, Norbert. I am a mathematician. The later life of a prodigy.** Doubleday and Co., Garden City, N.Y., 1956. 380 pp.

Kolmogorov, A. N.; and Stečkin, S. B. Sergeĭ Mihaĭlovič Nikol'skiĭ. (On his 50th birthday.) Uspehi Mat. Nauk (N.S.) 11 (1956), no. 2(68), 239–244 (1 plate). (Russian)

A list of Nikol'skiĭ's published papers is included.

Frank, Ludvík. On the life of Professor Mathias Lerch. Časopis Pěst. Mat. 78 (1953), 119–137. (Czech)

Škrášek, Josef. List of works of Prof. Mathias Lerch. Časopis Pěst. Mat. 78 (1953), 139–148. (Czech)

have ★**The collected works of George Abram Miller. Vol. IV.** University of Illinois, Urbana, Ill., 1955. xi+458 pp. $7.50.

Ninety eight of the 232 papers published by G. A. Miller during the period 1916 to 1929 are included in this volume prepared by H. R. Brahama, and the rest are listed in an appendix. Several of the papers are concerned with characteristic subgroups of finite groups, and several with substitution groups having certain properties. Three papers are concerned with groups generated by two elements of order three whose product is of order four or three or six, respectively, and others are concerned with groups having a small number of classes, or of Sylow subgroups, or of square operators, or a large number of elements of order 2. Four papers including the author's presidential address to the Mathematical Association of America in 1922 call attention to contradictions and inaccuracies in the literature, and others deal with points of historical interest or with mathematics and scientific research. [The second volume covering 1900 to 1907 was published in 1939 and reviewed in MR 1, 43; the third volume covering 1908 to 1915 was published in 1946 and reviewed in MR 8, 435]. *J. S. Frame.*

Whittaker, J. M. Obituary: Frederick Wrake Bradley. J. London Math. Soc. 31 (1956), 251–252.

Vint, J. Obituary: Henry Ronald Hassé. J. London Math. Soc. 31 (1956), 252–255.

LeBeau, G. S. Obituary: W. F. S. Churchill. J. London Math. Soc. 31 (1956), 256.

See also: Obláth, p. 1055.

FOUNDATIONS, MATHEMATICAL LOGIC

Destouches, J. L. Über den Aussagenkalkül der Experimentalaussagen. Arch. Math. Logik Grundlagenforsch. 2 (1956), 104–105.

Löb, M. H. Formal systems of constructive mathematics. J. Symb. Logic 21 (1956), 63–75.

Following the ideas of Myhill [same J. 15 (1950), 185–196; MR 12, 579] and of a previous paper of the author [ibid. 18 (1953), 1–6; MR 14, 938] a system of constructive arithmetic is developed. The first novelty is the omission of conjunction and of alternation from the list of primitives; they become definable by means of two individuals a and b (intuitively 0 and 1), concatenation, identity, existential and limited universal quantifiers. Secondly, it is shown that every statement-matrix of the system is expressible by another in the form of recursively enumerable predicates: $(E\alpha_1)(\beta)\alpha_1(E\alpha_2)\cdots(E\alpha_n)(\gamma=\delta)$, where α_1 limits the range of β to the chains with length equal or less than that of α_1. Finally, another system is constructed; it uses the μ-operator which here is interpreted so that $\mu_a\phi$ is the first string among the strings that satisfy ϕ. The ordering relation, as well as the rest of the apparatus is defined by means of only four primitives: μ, b, $=$, and concatenation. This system is shown to be equivalent to the first one in the sense of mutual definability of ideas. It is argued that constructive ideas (in the sense of Myhill) may be characterized as those expressible in such systems. *H. Hiz.*

Sanchez-Mazas, Miguel. Formalization of logic according to the perspective of intensionality. Theoria 3 (1955), no. 9, 105–117. (Spanish. French and English summaries)

The author investigates the classical syllogism formally from the point of view of intensional, as opposed to extensional, logic. The system used is part of the propo-

sitional calculus, with a new predicate, a new operation, and new classes of terms adjoined. All the laws of the logical square and the 24 syllogistic moods are demonstrated.

A class of nominal variables (a, a', b, b', $\cdots$) is introduced with the predicate '( )': $a(\,)b$ is interpreted to mean that a is incident with b. Defined are: $a)b$ [a includes b], and the negations of these two, $a)\,(b$, and $a(b$. If $a)\,(a'$, then a/a' is a new term. The four categorical propositions are defined in terms of the predicate ( ). For example, if $\mathbf{a}=a/a'$ and $\mathbf{b}=b/b'$, then $\mathbf{Aab}$ [All $\mathbf{a}$ is $\mathbf{b}$] means $a)b$ and $a')b'$.

It is especially interesting that this system can be interpreted not only in the above way, but also as the system of natural numbers under the relation of divisibility. Thus, if a and a' are natural numbers, a/a' is their quotient, and $a(\,)a'$ says that they are relatively prime. Under this interpretation the theorems of the system become number theoretic theorems concerning divisibility.
E. J. Cogan (Hanover, N.H.).

Copi, Irving M. Another variant of natural deduction. J. Symb. Logic **21** (1956), 52–55.

The author alters the conditions on variables in the Rule of Universal Generalization, as stated in his "Symbolic logic" [Macmillan, New York, 1954; MR **17**, 223] and proves the consistency of the new system of rules.
A. Heyting (Amsterdam).

Klaua, Dieter. Systematische Behandlung der lösbaren Fälle des Entscheidungsproblems für den Prädikatenkalkül der ersten Stufe. Z. Math. Logik Grundlagen Math. **1** (1955), 264–270.

This paper is concerned with the decision problem for the satisfiability of a prenex sentence of the lower predicate calculus (l.p.c.) whose prefix consists of two universal quantifiers followed by a number of existential quantifiers. Several solutions of the problem are known, and the present author gives a new derivation of the procedure of Schütte [Math. Ann. **109** (1934), 572–603]. The method of "symbolism resolution", which is used by the author, depends on the reduction of a sentence of the l.p.c. to an equivalent set of unquantified sentences (generally infinite) in a manner similar to that used by K. Gödel for the proof of the completeness of the l.p.c. [Cf. Church, Rev. Philos. Louvain **49** (1951), 203–221; Schröter, Z. Math. Logik Grundlagen Math. **1** (1955), 37–86; MR 17, 814]. *A. Robinson* (Toronto, Ont.).

Asser, Günter. Das Repräsentantenproblem im Prädikatenkalkül der ersten Stufe mit Identität. Z. Math. Logik Grundlagen Math. 1 (1955), 252–263.

Let X be a sentence of the lower predicate calculus with identity and let N be a set of positive integers such that $n\in N$ if and only if there exists a model of X which contains exactly n elements. In this case N is said to be representable. The present paper is concerned with the characterisation of all representable sets. A rather intricate necessary and sufficient condition is stated for arithmetical function $X(n)$ to be the characteristic function of a representable set. The condition shows that such a function is elementary in the sense of Kalmar [Cf. Kleene, Introduction to metamathematics, Van Nostrand, New York, 1952, p. 285; MR **14**, 525]. On the other hand, the author establishes that there exist nonrepresentable sets whose characteristic function is elementary. Examples of representable sets (some of which are by no means obvious) are given without proof and the author suggests that further research in this field is desirable. *A. Robinson* (Toronto, Ont.).

Zykov, A. A. The spectrum problem in the extended predicate calculus. Amer. Math. Soc. Transl. (2) 3 (1956), 1–14.

Translated from Izv. Akad. Nauk SSSR. Ser. Mat. 17 (1953), 63–76; MR **14**, 936.

Prior, A. N. Modality and quantification in S5. J. Symb. Logic 21 (1956), 60–62.

The author shows that, in the predicate calculus corresponding to S5, the formula $CM\Sigma x\phi x\Sigma xM\phi x$ is a theorem. This formula expresses the statement "If possibly something ϕ's, then something possibly ϕ's".
A. Rose (Nottingham).

Moch, François. La logique des attitudes. C. R. Acad. Sci. Paris **242** (1956), 1943–1945.

Moch develops a kind of many-valued logic in which a number of propositions not accepted by intuitionists are theorems. In a certain sense proof by reductio ad absurdum is permitted and applications to physical theories are discussed. *A. Rose* (Nottingham).

Schröter, Karl. Methoden zur Axiomatisierung beliebiger Aussagen- und Prädikatenkalküle. Z. Math. Logik Grundlagen Math. 1 (1955), 241–251.

This is a comparative study of several methods which are available for the deductive (axiomatic) development of calculi with two or more truth values. The author concentrates chiefly on calculi of propositions although some of his remarks apply also to functional calculi. Three lines of approach are distinguished, (i) the (most familiar) approach which depends on a theorem of substitution for equivalent propositions and on the reduction to a normal form, (ii) the step-by-step reduction of variables, and (iii) the explicit formulation of the relation of deducibility, after the manner of Gentzen's calculus. The two last mentioned methods are applied to the n-valued proportional calculus of Łukasiewicz. The author states that, as developed by him, the method applies equally well to certain first order functional calculi and to proportional calculi with an infinite number of truth values and promises a further paper on the subject. *A. Robinson.*

★**Markov, A. A. Teoriya algorifmov. [The theory of algorithms.]** Trudy Mat. Inst. Steklov. no. **42**. Izdat. Akad. Nauk SSSR, Moscow, 1954. 375 pp. 18.80 rubles.

The subject matter of this book has been previously presented in a series of notes [cf. Mal'cev, MR **2**, 7; Markov, MR **8**, 558; **9**, 221; **12**, 661; **13**, 4, 97, 811; **14**, 233; **16**, 436; Novikov, MR **14**, 618] of such fragmentary character that it was difficult for reviewers to feel sure of the correctness of the assertions made. The present book goes to the other extreme, and presents the subject in such full detail that in places it is tedious reading. Since it represents a major contribution to the logic of constructive processes, it will be reviewed here starting afresh.

By an algorithm the author means a constructive uniquely specified process. It may be applicable to various choices of the data; but when this choice is once made, every succeeding step is uniquely determined. Such a notion is involved in decision processes, in establishing constructive correspondences (homomorphisms) etc.

The author describes a particular sort of algorithm called a normal algorithm. In this the data consist of a certain "word" (i.e. finite string of letters) P in an alphabet A. The steps are specified by a finite list of instructions in a definite order; each instruction is of the form

$$X \to Y$$

where X and Y are particular words in A. If the algorithm $\mathfrak{A}$ has converted P to Q, the next step is determined as follows: one finds the first instruction in the list such that X occurs in Q; then one replaces the first occurrence of X by Y. It is provided that either X or Y may be void; if Y is void, the first occurrence of X is omitted from Q; if X is void, Q becomes YQ. To allow termination of the algorithm at an arbitrary point certain instructions are provided with a dot to indicate that the algorithm stops after executing, that instruction; the algorithm proceeds until either we reach such an instruction or a point where no instruction is applicable. It may, of course, happen that the process continues indefinitely, in which case we say the algorithm is not adapted to P; when the process terminates the result is called $\mathfrak{A}(P)$.

The book consists of two main parts. The first part (Chaps I–IV) contains a general theory of normal algorithms; the second part (Chaps. V–VI) applies this theory to show the impossibility of normal algorithms which answer certain decision questions.

In the first part the author explains the basic notions related to a normal algorithm, gives a number of examples of such algorithms, and adduces heuristic evidence for the thesis that any algorithm is equivalent to a normal algorithm. Given algorithms $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, etc., he shows how to construct derived algorithms, e.g. $\mathfrak{H}_1, \mathfrak{H}_2, \mathfrak{H}_3, \mathfrak{H}_4$, where

$$\mathfrak{H}_1(P) = \mathfrak{B}(\mathfrak{A}(P)) \qquad \text{(composition)}$$

$$\mathfrak{H}_2(P) = \mathfrak{A}(P) \wedge \mathfrak{B}(P) \qquad \text{(union)}$$

$$\mathfrak{H}_3(P) = \begin{cases} \mathfrak{A}(P) \text{ if } \mathfrak{C}(P) \text{ is void} \\ \mathfrak{B}(P) \text{ if } \mathfrak{C}(P) \text{ is not void} \end{cases} \qquad \text{(branching)}$$

while $\mathfrak{H}_4$ iterates $\mathfrak{A}$ until it reaches a Q, such that $\mathfrak{C}(Q)$ is void. (Here "$\wedge$" is used as symbol for concatenation; and it is understood that if either side of an equation exists, the other does also.) He considers the possibility of translating an algorithm from one alphabet into another. Finally he constructs a "universal" algorithm $\mathfrak{U}$ as follows: given any algorithm $\mathfrak{A}$, the specifications of $\mathfrak{A}$ can be written as a word $\mathfrak{A}^*$ in a suitable alphabet; then for all P

$$\mathfrak{U}(\mathfrak{A}^* P) = \mathfrak{A}(P).$$

In one variant of this process $\mathfrak{A}^*$ and $\mathfrak{U}$ can be expressed in the same alphabet as $\mathfrak{A}$.

In the second part the author introduces the related notions of associative calculus and Post calculus. In the former we have a list of replacements with no assigned order; the replacement can be made in either direction; derivability is thus an equivalence relation, and we have an algebra with concatenation as an associative operation. In the latter we have rules of transformation of the form

$$XP \to PY,$$

where P is an arbitrary word and X, Y are given words; such a rule is applicable to Q if and only if XP is the whole of Q. Then among the problems which the author shows are undecidable by a normal algorithm are the following: the applicability of an argument algorithm to its own description; the applicability of certain algorithms to an argument word, and the detection of those words which certain algorithms annul; the completeness (polnota) of an argument algorithm, i.e. its applicability to a certain description $\mathfrak{A}^*$ of any algorithm $\mathfrak{A}$; the equivalence (intertransformability) of two given words in certain associative calculuses; the deducibility of a word in certain Post calculuses with a given initial word; the solution (under certain conditions) of a combinatorial problem of Post [Amer. J. Math. **65** (1943), 197–215; MR **4**, 209]; certain problems in the representation of matrices of natural numbers as products; and problems of ascertaining whether a calculus has a given property, provided the property is invariant of isomorphism and there exist calculuses which have the property as well as others which are not isomorphic to any part of a calculus having the property. *H. B. Curry.*

Kuznecov, A. V.; and Trahtenbrot, B. A. Investigation of partially recursive operators by means of the theory of Baire space. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 897–900. (Russian)

This study is motivated by results of two papers: E. Post, Bull. Amer. Math. Soc. **50** (1944), 284–316 [MR **6**, 29] from which follows the existence of a pair of recursively enumerable sets E_1 and E_2 such that E_1 does not reduce to E_2 by means of any general recursive operator; and J. C. E. Dekker, Proc. Amer. Math. Soc. **5** (1954), 791–796 [MR **16**, 209] in which is shown that to every recursively enumerable set E_1 there exists a hypersimple set E_2 to which E_1 reduces by means of a partial recursive operator. The authors investigate partial recursive operators $g=T[f]$, where f is a function of one argument, and g either a constant or a function of one argument. More general situations may be reduced to this one.

Each function $f(x)$ is represented as a point (the ω-tuple $\langle f(0), f(1_n), \cdots, f(n), \cdots\rangle$) in Baire space J. The Baire intervals δ are enumerated primitive recursively. Every partial recursive operator $g=T(f)$ considered on J can be represented as $g(x)=b(\mu t(f \in \delta^{a(x,t)}))$, where a and b are primitive recursive functions. Based on ordinary Baire distance topological notions are defined: effective continuity, effective compactness. Certain sets are identified as effectively closed, effectively open, effectively G_δ and effectively F_σ. Properties of partial recursive operators are discussed in terms of these notions. *E. J. Cogan* (Hanover, N.H.).

Myhill, J.; and Shepherdson, J. C. Effective operations on partial recursive functions. Z. Math. Logik Grundlagen Math. **1** (1955), 310–317.

Defining a completely recursively enumerable family of partial recursive functions by analogy with a class of recursively enumerable (c.r.e.) sets [Rice, Trans. Amer. Math. Soc. **74** (1953), 358–366; MR **14**, 713], the authors give a direct proof of the structure theorem for c.r.e. families: such a family consists of all partial recursive extensions of the explicitly given finite (i.e., finite domain) functions in a recursively enumerable array, and conversely. From this follows directly the corresponding fact for c.r.e. classes of sets. Calling an operation (transformation) on the recursively enumerable sets effective when from a Gödel number of a set one effectively obtains a Gödel number of the image set, a normal form for effective operations on sets is obtained, and several examples given.

The main result concerns effective operations on partial

recursive functions, defined analogously. It is well-known that a partial recursive functional [Kleene, Introduction to metamathematics, Van Nostrand, New York, 1952; MR 14, 525] is such an operation; the authors here establish the converse, using the theory of c.r.e. families developed above. The paper closes with a discussion of some unanswered questions, especially concerning operations effective on some subfamily of the partial recursive functions, and analogous problems for sets. *H. G. Rice* (Pittsburgh, Pa.).

Molodšiĭ, V. N. On interrelations of certain assertions of generality with the induction axiom in Peano's system of axioms. Moskov. Gos. Univ. Uč. Zap. 155, Mat. 5 (1952), 168–173. (Russian)

This paper concerns the relation of the system of Peano arithmetic to assertions $\bar{E}$ which are negations of statements E of the following form: $Em_1 \cdots Em_p \mathfrak{Y}(m_1, \cdots, m_p)$ where $\mathfrak{Y}$ is a conjunction or disjunction of statements that terms formed from $m_1, \cdots, m_p$, and the natural numbers by the operations $+$ and $\times$ are equal, and negations of such statements. In particular, it is shown that the induction axiom is not equivalent to any expression of the form $\bar{E}$.

This is accomplished by considering $P\&\bar{I}$, a system of axioms for a set of numbers which satisfy the first four Peano axioms, but not the induction axiom, and which contains a subset isomorphic in order to the natural numbers, but not exhausting them. The result is then a corollary of the main theorem: there exists a model for $P\&\bar{I}$ made up of ordered pairs of natural numbers, which pairs are ordered by a certain law, such that each expression $\bar{E}$ which holds for natural numbers holds also for the numbers of this model beginning with the Nth in the sequence. The number N depends on $\bar{E}$. The elements of the model are pairs (m, n) of natural numbers for which $m \geqq n$. These pairs are ordered lexicographically.

The author remarks that the result cannot be extended to apply to expressions E in which both universal and existential quantifiers appear. *E. J. Cogan.*

Goodstein, R. L. The Arabic numerals, numbers and the definition of counting. Math. Gaz. 40 (1956), 114–129.

Have

★ Kleene, S. C. Representation of events in nerve nets and finite automata. Automata studies, pp. 3–41. Annals of mathematics studies, no. 34. Princeton University Press, Princeton, N. J., 1956. $4.00.

The notion of a regular event is introduced in this paper via certain binary operations on sets of tables which describe some property of inputs to a nerve net. It is shown (Theorems 3 and 5) that there is a nerve net representation of each regular event and that in any finite automaton with a given initial state, the event represented by a given state existing at a later time is a regular event. An example of an irregular event is given.

The paper includes a brief account of the McCulloch-Pitts nerve net model and a discussion of various qualifications on events (definite, positive, initial, et cetera) and their logical ramifications. *C. Y. Lee* (Chatham, N.J.).

Have

★ von Neumann, J. Probabilistic logics and the synthesis of reliable organisms from unreliable components. Automata studies, pp. 43–98. Annals of mathematics studies, no. 34. Princeton University Press, Princeton, N. J., 1956. $4.00.

In synthesis of automata (animal and physical mechanisms), the role of error is treated in this paper as an integral part of the synthesis process. The author gives a constructive procedure for synthesizing an automaton whose probability of malfunction in the final output is not to be greater than η with components with known reliability ε, where ε is assumed to be small to begin with. This procedure is carried out explicitly for systems synthesized with sheffer stroke organs and implicitly for systems synthesized with other organs. For systems using sheffer organs, this procedure consists of finding, through a statistical treatment, a number N depending on η and ε, and showing that by multiplexing the lines N times and the organ $3N$ times (N times for executive function and $2N$ times for restoring function), the automaton can be made to behave with $1-\eta$ perfection.

In addition to the multiplexing procedure, a triplication procedure is also described. Although error can be controlled by the triplication or non-multiplexing procedure as well, it is pointed out that multiplexing is superior in that far fewer components are required, especially for automata with a fair amount of "logical depth".

Apart from the central theme, numerous other topics are touched upon and illuminating remarks given. It is shown as a step in synthesis that the logical and, or and not functions can be made to have equal time delay if a double line representation is used (as in the case of flip-flops). Possible resemblance of multiplexing in animal nervous system is discussed. The paper concludes with some remarks on analog (as contrasted with digital) model of neuron behavior. *C. Y. Lee* (Chatham, N.J.).

See also: Russell, p. 1121; Février, p. 1161.

ALGEBRA

Have

★ Cartan, Henri; and Eilenberg, Samuel. Homological algebra. Princeton University Press, Princeton, N. J., 1956. xv+390 pp. $7.50.

The title "Homological Algebra" is intended to designate a part of pure algebra which is the result of making algebraic homology theory independent of its original habitat in topology and building it up to a general theory of modules over associative rings. The particular formal aspect of this theory stemming from algebraic topology is that of a preoccupation with endomorphisms of square 0 in graded modules. The conceptual flavor of homological algebra derives less specifically from topology than from the general 'naturalistic' trend of mathematics as a whole to supplement the study of the anatomy of any mathematical entity with an analysis of its behavior under the maps belonging to the larger mathematical system with which it is associated. In particular, homological algebra is concerned not so much with the intrinsic structure of modules but primarily with the pattern of compositions of homomorphisms between modules and their interplay with the various constructions by which new modules may be obtained from given ones.

In the recent requisite mathematical terminology, this means that the principal objects of study in homological algebra are functors from categories of modules to other categories of modules. If R and S are two rings (with identity elements that act as the identity on all modules considered) a covariant functor (of one variable) from the

category of R-modules to the category of S-modules is a function T attaching to each R-module A an S-module $T(A)$, and to each R-homomorphism h: $A'\to A$ an S-homomorphism $T(h)$: $T(A')\to T(A)$ such that, if k is a second homomorphism $A\to A''$, $T(k\circ h)=T(k)\circ T(h)$ is the identity map $T(A)\to T(A)$ whenever h is the identity map $A\to A$. For a contravariant functor, one has instead $T(h)$: $T(A)\to T(A')$, and $T(k\circ h)=T(h)\circ T(k)$. The definition of a functor in several variables in different categories, and some covariant, some contravariant, is the evident extension of the above, with an additional commutativity requirement for maps acting on arguments in different positions in the functor. The functor T is said to be additive if, for any two homomorphisms h_1 and h_2 between the same modules, one has $T(h_1+h_2)=T(h_1)+T(h_2)$. As a linear theory, homological algebra deals only with additive functors.

A functor is called exact if (for each variable separately) it maps every exact sequence into an exact sequence. It is the essence of homological algebra to channel any deviation of a functor from exactness into the construction of new functors. For these constructions, the following notions are basic. A module M is called projective if every homomorphism of M into a factor module A/B can be lifted to A; M is called injective if every homomorphism of B into M can be extended to A. Every module is a homomorphic image of a projective module and can be imbedded in an injective module. Hence every module A has a "projective resolution", i.e., an exact sequence $\cdots\to X_n\to\cdots\to X_0\to A\to 0$ in which the X_i are projective, and an "injective resolution", i.e., an exact sequence $0\to A\to X^0\to\cdots\to X^n\to\cdots$ in which the X^i are injective. If projective resolutions are substituted for all the contravariant variables in a functor and injective resolutions are substituted for all the covariant variables a complex is obtained whose homology groups are, up to natural isomorphisms, independent of the particular choice of the resolutions. The nth homology group of this complex is the module value of the nth right derived functor R^nT of the given functor T. Similarly, one obtains the left derived functors L_nT by substituting injective resolutions for the contravariant variables and projective resolutions for the covariant variables. These derived functors have the same variance as T, and vanish for $n<0$.

The derived functors are related by the so-called connecting homomorphisms which increase the degree by 1 in the case of the right derived functors and lower the degree by 1 in the case of the left derived functors. These result from any exact sequence $0\to A'\to A\to A''\to 0$ when A', A, A'' are substituted in turn for any one of the variables. If the functor is covariant in this variable the connecting homomorphism sends $R^nT(A'')$ into $R^{n+1}T(A')$ and, with the usual functor homomorphisms, forms an exact sequence

$$\cdots\to R^nT(A')\to R^nT(A)\to R^nT(A'')\to R^{n+1}T(A')\to\cdots.$$

The case of contravariance, and the case of the left derived functors, are obtained by making the evident appropriate changes.

A covariant functor T of one variable is said to be left exact if, for every exact sequence $0\to A'\to A\to A''\to 0$, the sequence $0\to T(A')\to T(A)\to T(A'')$ is exact. In case T is contravariant, interchange A'' and A' in the last sequence. For functors of several variables, one demands that this property hold for each variable. One says that T is right exact if, instead of the above, the sequence $T(A')\to T(A)\to T(A'')\to 0$ is exact, etc. If T is left exact then the left derived functors L_nT are 0 (except for $n=0$) and R^0T may be identified with T. If T is right exact the right derived functors R^nT are 0 (except for $n=0$) and L_0T may be identified with T.

A functor T is said to be right balanced if substitution of any projective module for a contravariant variable, or substitution of any injective module for a covariant variable, yields an exact functor in the remaining variables. The definition of a left balanced functor is obtained by interchanging the words projective and injective. Much of the power of the mechanism of derived functors results from the following facts: if T is right balanced then its full right derived functors R^nT may be identified with the partial right derived functors obtained by using resolutions for any non-empty subset of variables and treating all the other variables as constants in the construction; if T is left balanced then the same result holds for the left derived functors L_nT.

The above basic principles and results are established in chapters I–V. Chapter VI concentrates on the two basic functors of homological algebra and their derived functors. The first basic functor is the twice covariant functor $\otimes_R$ that attaches to each pair (A, B), where A is a right R-module and B a left R-module, the tensor product $A\otimes_R B$, regarded as a module over the ring Z of the integers, in general. This functor is right exact and left balanced. The second basic functor is the contravariant-covariant functor Hom_R that attaches to each pair (A, B) of left R-modules the Z-module $\mathrm{Hom}_R(A, B)$ of all R-homomorphisms of A into B. This functor is left exact and right balanced. The left derived functors of $\otimes_R$ are denoted $\mathrm{Tor}_n{}^R$, and the right derived functors of Hom_R are denoted $\mathrm{Ext}_R{}^n$.

Some of the simpler interpretations and applications of these functors are given in Chapters VI and VII. An R-module A is said to be of projective dimension $\leqq n$ if it has a projective resolution $\cdots\to X_i\to\cdots\to X_0\to A\to 0$ such that $X_i=0$, for all $i>n$. There is a similar, but less frequently used, notion of injective dimension. One says that a ring R has left global dimension $\leqq n$ if every left R-module has projective dimension $\leqq n$ or — equivalently — if $\mathrm{Ext}_R{}^{n+1}=0$. Another equivalent condition is that every left R-module be of injective dimension $\leqq n$. The significance of this notion for ring theory is already indicated by the following facts: the rings of global dimension 0 are precisely the semisimple rings, i.e., the rings R such that every R-module is a direct sum of simple submodules; a ring has left global dimension $\leqq 1$ if and only if every left ideal is a projective module; an integral domain has this property if and only if it is a Dedekind ring, i.e., if and only if the classical ideal theory holds for the ring.

The functor $\mathrm{Tor}_1{}^R$ plays an important role in the Künneth relations between the homology groups of a tensor product of complexes and those of the factors. In fact, it arose for the first time exactly in this context. Here, the Künneth relations are obtained in full group theoretic form and under far more general conditions than those obtaining in the original topological situation of the integral homology groups of product spaces.

Chapter VIII introduces the homology theory of augmented rings from which the specific homology (and cohomology) theories of associative algebras, groups, and Lie algebras are obtained by specialization. The structure of an augmented ring consists of a ring R and an R-epimorphism of R onto an R-module Q. The nth homology group of the augmented ring R in a right R-module A is

defined as $\mathrm{Tor}_n{}^R(A, Q)$. The nth cohomology group of R in a left R-module A is defined as $\mathrm{Ext}_R{}^n(Q, A)$. These notions apply rather directly to a dimension theory for local rings and graded rings yielding, for instance, various generalizations and extensions of Hilbert's theorem on chains of syzygies of homogeneous polynomial ideals.

Chapter IX discusses the homology and cohomology theory of associative algebras L (with 1) over commutative rings K (with 1). Define L^e as the tensor product algebra $L\otimes_K L'$, where L' denotes the usual anti-isomorph of L. Together with the L^e-epimorphism $L^e\to L$ that sends $a\otimes b'$ onto ab. this is an augmented ring structure. By using a suitable L^e-projective resolution of L, it is shown that the usual cohomology groups $H^n(L, A)$ of L in an L^e-module A are nothing but the groups $\mathrm{Ext}_{L^e}{}^n(L, A)$. This makes the general theory available for the study of the cohomology of algebras and thus gives a high degree of control over it. Of particular interest is the cohomological dimension of L which is defined to be $\leqq n$ if and only if $H^{n+1}(L, A)=0$ for all L^e-modules A.

The homology theories for groups and Lie algebras have certain special features in common, and Chapter X introduces the requisite pre-specialization of the augmented ring theory. The basic structure is that of a supplemented algebra. It consists of a K-algebra L, together with a unitary algebra epimorphism of L onto K. Close relations exist between the homology of L as a supplemented algebra and the homology of L^e, augmented as above. In many situations, this leads to a strengthening of each theory. The homology and cohomology groups of a group G are obtained from the supplemented algebra structure consisting of the group algebra $Z(G)$ and the coefficient sum epimorphism $Z(G)\to Z$.

Chapter XI gives the general foundation for the multiplicative theory. Four "external products" are defined, involving two K-algebras L and M, their tensor product $L\otimes_K M$, and the functors Tor and Ext for these three algebras. "Internal products", involving only one algebra, can be derived from these with the aid of algebra homomorphisms $L\otimes_K L\to L$ or $L\to L\otimes_K L$. In particular, in the case of groups or Lie algebras, one has natural homomorphisms $L\to L\otimes_K L$ by means of which the cup and cap products are derived from two of the external products.

The most complete and most powerful cohomology theory is that of a finite group G. Chapter XII develops the special formal machinery that governs this case. The main point (discovered by J. T. Tate in connection with the application to class field theory) is that the norm map in G-modules ($N(a)=\sum_{x\in G} x\cdot a$)leads to a tie-up between homology and cohomology by which the homology groups play the role of cohomology groups of negative degree and link up with the usual cohomology groups to form the so-called complete derived sequence of G. The connecting homomorphisms go through this complete sequence, and a product can be defined which extends simultaneously the cup and the cap product and applies without restriction to arbitrary members of the complete derived sequence.

Chapter XIII brings the homology and cohomology theory of Lie algebras into the framework of supplemented algebras. The critical device for doing this is to pass to the universal enveloping algebra; with every Lie algebra L one can uniquely associate an associative algebra L^e in such a way that there is a natural 1-1 correspondence between the representations of L and those of L^e. Assuming that L is free over the ground ring K, one has $L\subset L^e$, and the commutation in L is given by the multiplication in L^e:$[x, y]=xy-yx$. There is a natural algebra epimorphism of L^e onto K whose kernel coincides with the subalgebra of L^e that is generated by L. This is the supplemented algebra structure whose homology and cohomology is that of L. It is, a priori, not at all evident that this actually yields the theory as originally formulated under the inspiration of the differential geometry on Lie groups. The identification requires the use of a certain L^e-projective resolution of K involving the exterior algebra built over L. Interesting by-products are obtained concerning L^e. The cohomological dimension of L^e coincides with the projective dimension of K as an L^e-module. If K is semisimple the cohomological dimension of L^e coincides with its right and left global dimensions. In particular, if L has rank n over K, these dimensions are all equal to n. This generalizes earlier results on polynomial algebras, because if L is abelian then L^e is simply $K[x_1, \cdots, x_n]$.

Chapter XIV takes up the applications of cohomology theory to various extension theories. The simplest case is that of extensions of R-modules. To each such extension $0\to C\to X\to A\to 0$ one can associate an element of $\mathrm{Ext}_R{}^1(A, C)$ which is 0 if and only if the extension is split. This correspondence yields an isomorphism of the group of equivalence classes of the extensions of C by A (with the Baer composition) onto $\mathrm{Ext}_R{}^1(A, C)$. This is actually the origin of the notation Ext. The other three cases are extensions of K-projective K-algebras with kernels of square 0, group extensions with abelian kernels, and Lie algebra extensions with abelian kernels. In each of these cases one obtains an interpretation of the second cohomology group as a group of equivalence classes of extensions.

Chapter XV gives a complete development of the formalism of spectral sequences. In particular, two spectral sequences are associated with every double complex. The terms E_∞ of these spectral sequences are the graded modules associated with the homology module by the two natural filtrations of the double complex. The terms E_2 of the spectral sequences are the compound homology modules obtained from the double complex by using the two differential operators in succession. In Chapter XVI it is shown how these spectral sequences can be used for establishing connections between partial derived functros and full derived functors, and for investigating the behavior of Tor and Ext under homomorphisms of operator rings. It is also shown how the general method yields the spectral sequences that have been used for examining the relations between the homology and cohomology of a group or a Lie algebra and that of a normal subgroup or an ideal. The rest of Chapter XVI sketches some topological applications concerning spaces with groups of operators.

Chapter XVII reaches a very high degree of complexity. Resolutions of complexes take the place of resolutions of modules, and spectral sequences are obtained which connect the results of first applying a functor and then passing to homology with the results of first passing to homology and then applying the functor. In particular, this leads to a fuller and more general treatment of the Künneth relations.

The appendix by D. A. Buchsbaum proposes an abstract framework of "exact categories" that is capable of accommodating the functor theory of this book as well as additional structural elements that one may wish to introduce. The proposed theory includes an abstract notion of duality which makes it unnecessary, at least in principle, to give separate treatments for covariance

and contravariance and for projectivity and injectivity.

The appearance of this book must mean that the experimental phase of homological algebra is now surpassed. The diverse original homological constructions in various algebraic systems which were frequently of an ad hoc and artificial nature have been absorbed in a general theory whose significance goes far beyond its sources. The basic principles of homological algebra, and in particular the full functorial control over the manipulation of tensor products and modules of operator homomorphisms, will undoubtedly become standard algebraic technique already on the elementary level. It is probably with such expectations that the authors have put so much missionary zeal into the systematization of their approach and the cataloguing of the basic results. A probably unavoidable effect of this is that the book cannot be consulted by the uninitiated in a local fashion. The reader is definitely forced to go through it starting at the beginning. Each chapter (with the exception of the last) is followed by a collection of exercises which are designed not so much to strengthen the reader by easy gymnastics (for they are generally not particularly easy) as to point out various ramifications and applications of the general theory.

A few misprints might lead to confusion: p. 80, in the diagram above Prop. 2.3, the top X should have a double prime and the X's in the bottom row should have single primes; p. 101, first line below Lemma 10.1, the middle term of the sequence should have a double prime; p. 152, lines 4 and 7, replace M/J_k by M/MJ_k and M/J_{k-1} by M/MJ_{k-1}; p. 233, diagram (1), reverse the arrow labelled g; p. 319, line 5 from bottom, replace d_r by d_k; p. 322, third line above Theorem 3.2, replace dsx by sdx.

G. Hochschild (Berkeley, Calif.).

★**Monjallon, Albert. Initiation au calcul matriciel. Matrices — déterminants. Applications à l'algèbre et à la géométrie analytique.** Librairie Vuibert, Paris, 1955. 131 pp. 700 francs.

The purpose of this book is to teach the reader the most important facts about vectors, matrices and determinants without necessarily giving complete proofs of the underlying theorems. The author often confines himself to stating and describing the theorems clearly and then illustrating them with numerical examples. Thus, a reader who is mainly interested in the applications of matrix algebra can acquire a particular technique without knowing why it works. The book also serves as an introduction to linear algebra for young students or pupils who would find the algebraical theory too advanced. As an educational device this policy has much to be commended; for if well done — and the present volume is written with great care and considerable expository skill — the learner's enthusiasm might be aroused by handling new tools whose power, though somewhat mysterious, is obvious even to a beginner.

The topics covered are the formal laws of matrix algebra, the simplest properties of determinants (with proofs), systems of linear equations, reduction of quadratic forms both by congruent and by orthogonal transformation and some geometrical discussion of coordinate transformations.

There are a few places where the reviewer felt that a little more explanation would be more in keeping with the very elementary character of the book: (i) the remarks about repeated latent roots might erroneously be taken to imply that the diagonalizing process will always fail in this case; (ii) it is not made sufficiently clear how the congruent reduction of a quadratic form is carried out in general; (iii) the brief mention of a group might mislead a reader who is not familiar with this concept, since the group axioms are not stated in full.

The attractively written book will be welcomed by the class of readers for whom it is intended. *W. Ledermann.*

Suškevič, A. K. An algebra defined as an infinite direct sum of rings. Har'kov. Gos. Univ. Uč. Zap. 40=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) 23 (1952), 49–60 (1954). (Russian)

Various properties of rings of the type mentioned are expounded. A limit process for such a direct sum can be defined as follows: $\lim_n A_n = A$ means that eventually A_n agrees with A in the first M components. Except for the absence of examples based on specific rings of numbers, the exposition has didactic merit. *J. L. Brenner.*

Suškevič, A. K. On an infinite algebra of triangular matrices. Har'kov. Gos. Univ. Uč. Zap. 34=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) 22 (1950), 77–93 (1951). (Russian)

The (upper) triangular matrices form a ring. There is such a matrix Z in which infinitely many diagonal elements are 0, but the equation $XZ=0$ has no solution X $(\neq 0)$. Collections of matrices are defined which form, and which fail to form, ideals, as the case may be.

J. L. Brenner (Pullman, Wash.).

Suškevič, A. K. On a type of algebras of infinite matrices. Har'kov. Gos. Univ. Uč. Zap. 29=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) 21 (1949), 131–144. (Russian)

The algebra considered is the collection of all matrices which are both row-bounded and column-bounded. Certain ideals and ring-theoretic decompositions are given. *J. L. Brenner* (Pullman, Wash.).

Lunc, A. G. Reduction of a matrix to Jordan normal form. Leningrad. Gos. Univ. Uč. Zap. 144. Ser. Mat. Nauk 23 (1952), 35–46. (Russian)

This article establishes the existence and uniqueness of the Jordan normal form of a matrix over an algebraically closed field. The proof depends ultimately on the Hamilton-Cayley theorem and a few computational but not unduly intricate lemmas. Hence its validity extends to other domains in which the Hamilton-Cayley theorem holds. The author makes the point of avoiding the idea of linear transformation. — The first step is to give a direct proof that the matrix equation $AX-XB=C$ has a unique solution if A, B have no common root. — The general treatment of the subject is valuable on the whole, and noteworthy in that it was completed by 1949.

J. L. Brenner (Pullman, Wash.).

Combinatorial Analysis

Norton, D. A.; and Stein, Sherman K. An integer associated with Latin squares. Proc. Amer. Math. Soc. 7 (1956), 331–334.

On sait que tout carré latin de côté n est la table de Cayley d'un quasigroupe fini, $Q(\times)$, d'ordre n. Si, pour $x, y, z \in Q$ on a: $x\times y=z$ et si à chacun des n^2 éléments de l'ensemble produit $Q\times Q$ on fait correspondre le point de coordonnées x, y, z, en vertu de la loi du quotient, chaque

point est entièrement déterminé par deux de ses coordonnées. Si Q est idempotent on peut réaliser sur l'ensemble des points dont l'une des coordonnées est un nombre fixé m, une partition telle que deux points sont équivalents si l'on passe de l'un à l'autre en permutant deux de leurs coordonnées — la troisième restant déterminée par la loi de Q. Ex: $(m, y, m\times y)\equiv(m\times y, t, m)$ avec $(m\times y)\times t=m$. Si $m=y$; la classe est monome et se réduit au point (m, m, m). Le nombre des points des autres classes est divisible par 3. Soit $\phi(m)$ le nombre des classes non monomes définies par cette équivalence. Il est montré que $Z=\sum\phi(m)\equiv n(n-1)/2 \pmod 2$ quand m décrit Q. Observation du reviewer. 1) Il existe une large catégorie de quasigroupes pour lesquels, non seulement Z est de même parité que $n(n-1)/2$, mais lui est encore égal. Tels sont ceux où deux éléments quelconques engendrent toujours un diviseur du 3ème ordre [cf. Sade, Quasigroupes, Marseille, 1950, no 10, 11; MR 13, 203], ou encore ceux qui ont une loi de composition comme

$$R(\times): x\times y\equiv ax+(1-a)y \pmod n,$$

dont toutes les classes, non monomes, renferment exactement six points, propriété que R garde, d'ailleurs, s'il est défini sur l'ensemble des nombres réels, de sorte que $\phi(m)=(n-1)/2$ quel que soit m. 2) Si l'on regarde comme équivalents deux éléments m et m' tels que $\phi(m)=\phi(m')$, tout automorphisme de Q transformera chaque élément en un élément équivalent. La connaissance des classes, modulo ϕ, peut donc rendre des services dans la recherche de l'automorphe de Q. *A. Sade.*

Ward, James A. A certain bridge tournament seating problem and latin squares. Math. Mag. **29** (1956), 249–253.

The bridge tournament problem is the following: In how many ways may c married couples play $c-1$ rounds, if each man plays one round with each lady except his wife, and everybody plays against everybody else except his or her spouse? It is shown that a solution exists if there are three c by c latin squares which are orthogonal in pairs, and that for $c=2^k$ the number of solutions is $(2^k-1)(2^k-2)$; an explicit construction is given.

J. Riordan (New York, N.Y.).

Linear Algebra, Polynomials, Invariants

Brauer, Alfred Bounds for the ratios of the co-ordinates of the characteristic vectors of a matrix. Proc. Nat. Acad. Sci. U.S.A. **41** (1955), 162–164.

Let $Ax=\omega x$ with $|x_r|\geqq|x_\lambda|$ for $\lambda\neq r$ and R the circle $|z-a_{rr}|\leqq\sum_{\lambda\neq r}|a_{r\lambda}|\equiv P_r$ or any closed subregion of it containing ω. If $d_\kappa=\min$ (distance of $a_{\kappa\kappa}$ from R), $t_{r\kappa}^{(\nu)}=1$ if $d_\kappa=0$ or $t_{r\kappa}^{(\nu)}=\min(1, P_\kappa d_\kappa^{-1})$ and

$$t_{r\kappa}^{(\nu)}=\min\{1, d_\kappa^{-1}(|a_{\kappa r}|+\textstyle\sum_{\lambda\neq\kappa,r}|a_{\kappa\lambda}|t_{r\lambda}^{(\nu-1)})\},$$

then $|x_\kappa|\leqq t_{r\kappa}^{(\nu)}|x_r|$ for $\nu=2, 3, \cdots$ and $\kappa=1, 2, \cdots, n$. The root ω must also lie in $|z-a_{rr}|\leqq\sum_{\lambda\neq r}|a_{r\lambda}|t_{r\lambda}^{(\nu)}$.

W. Givens (Knoxville, Tenn.).

Brauer, Alfred; and LaBorde, H. T. Limits for the characteristic roots of a matrix. VI. Numerical computation of characteristic roots and of the error in the approximate solution of linear equations. Duke Math. J. **22** (1955), 253–261.

[For parts I–V see Brauer, same J. **13** (1946), 387–395; **14** (1947), 21–26; **15** (1948), 871–877; **19** (1952), 75–91, 553–562; MR **8**, 192, 559; **10**, 231; **13**, 813; **14**, 836.] For A a n by n and B a m by m submatrix of $\lambda I-A$, let R be a closed subregion of the complex number plane in which det B is known to have a zero $\lambda=\omega$ while B_{mm} is (after permutation of rows and columns) an $m-1$ by $m-1$ principal minor of B such that det B_{mm} is not zero in R and at least one of $b_{1m}, b_{2m}, \cdots, b_{m-1,m}$ is a non-zero constant. Let

$$G_\mu=\max_R\Big|\sum_1^{m-1}b_{\mu\nu}x_\nu^*+b_{\mu m}\Big|,\quad |G|=\Big[\sum_1^{m-1}G_\mu^2\Big]^{\frac12}$$

and $\eta=$minimum for λ in R of the smallest (necessarily positive) root of $B_{mm}^*(\lambda)B_{mm}(\lambda)$. Then $|x_\mu^*-x_\mu|\leqq|G|/\eta^{\frac12}$ when $(x_1, \cdots, x_{m-1}, 1)$ is an exact and $(x_1^*, \cdots, x_{m-1}^*, 1)$ is an approximate solution of $Bx=0$. For R a circle, the G_μ are calculated easily. Dropping only the assumption on det B, if no vector $(\xi_1, \cdots, \xi_{m-1}, 1)$ satisfying

$$|x_\mu^*-\xi_\mu|\leqq|G|/\eta^{\frac12}\quad(\mu=1, 2, \cdots, m-1)$$

satisfies the last of the equations $Bx=0$, then det B has no zero in R. The results are used to conclude that by "Solving linear and quadratic equations only it is possible to determine the different characteristic roots of A as exactly as necessary". A numerical example of order four is worked in detail. Since choices of minors must be made, information on roots of smaller minors used to obtain roots of larger minors and other combinatorial difficulties may cascade, the practical use in numerical computation seems limited to situations permitting human judgment. *W. Givens* (Knoxville, Tenn.).

Goldberg, Karl. A matrix with real characteristic roots. J. Res. Nat. Bur. Standards **56** (1956), 87.

The significant result is that if A is a matrix whose diagonal elements are non-negative and whose off-diagonal elements are non-positive, and if, for every k it is true that

$$a_{i_1i_2}a_{i_2i_3}\cdots a_{i_ki_1}=a_{i_1i_k}\cdots a_{i_3i_2}a_{i_2i_1},$$

then the characteristic roots of A are all real. The proof is made by showing equality of each principal minor of A to the corresponding minor of the matrix whose elements are $b_{ij}=-(a_{ij}a_{ji})^{\frac12}$. An additional condition which the author imposes upon A merely permits the roots to be bounded. *A. S. Householder* (Oak Ridge, Tenn.).

Khan, Nisar A. A theorem on the characteristic roots of matrices. J. Univ. Bombay. Sect. A. (N.S.) **24** (1955), no. 38, 13–18.

This paper has the same mathematical content as the paper by Roy [Proc. Amer. Math. Soc. **5** (1954), 635–638; MR **16**, 4], and the same remarks apply. *J. L. Brenner.*

Ville, J. Principes d'analyse matricielle. Publ. Inst. Statist. Univ. Paris **4** (1955), 141–217.

This paper gives an exposition of certain elementary topics in matric theory (43 pages) and in the theory of probability (27 pages). No references are cited.

M. F. Smiley (Iowa City, Ia.).

Hukuhara, Masuo. Théorie des endomorphismes de l'espace vectoriel. II. J. Fac. Sci. Univ. Tokyo. Sect. I. **7** (1956), 305–332.

Suite d'un article antérieur [même J. **7** (1954), 129–192; MR **16**, 992]. La terminologie est celle de cet article. L'A. étudie l'existence d'endomorphismes supplémentaires d'un endomorphisme donné et divers problèmes voisins.

Par exemple, un endomorphisme pour lequel

$$\min(\mu, \nu) < +\infty$$

admet un supplémentaire. Soient K un endomorphisme, $L(\lambda) = I - \lambda K$. Appliquant les définitions posées pour K à l'endomorphisme $L(\lambda)$, l'A. obtient divers sous-espaces dépendant de λ. Relations entre ces sous-espaces et les sous-espaces associés au transposé ou au supplémentaire éventuels de K. Développements de vecteurs en vecteurs propres. Développements de la résolvante de K.

J. Dixmier (Paris).

Lotze, A. Über eine neue Begründung der regressiven Multiplikation extensiver Grössen in einem Hauptgebiet n-ter Stufe. Jber. Deutsch. Math. Verein. **57** (1955), 102–110.

The paper deals with the formal calculus of the exterior algebra of a vector space, of its dual and especially of their interrelation. The notation and concepts are oriented toward Grassmann's Ausdehnungslehre rather than toward algebraic treatments such as Bourbaki's "Algèbre multilinéaire" [Actualités Sci. Ind., no. 1044, Hermann Paris, 1948; MR **10**, 231]. *W. Givens.*

Mitrinovitch, Dragoslav S. Sur le déterminant de Stern généralisé. Bull. Soc. Math. Phys. Serbie **7** (1955), 153–160 (1956). (Serbo-Croatian summary)

For n, k positive integers ($k \leqq n$), the determinant $D_{n,k}$ of order n considered in this paper has

$$1, \binom{r_i}{1}, \binom{r_i}{2}, \cdots, \binom{r_i}{k-1}, \binom{r_i}{k+1}, \cdots, \binom{r_i}{n}$$

as its ith row, $(i = 1, 2, \cdots, n)$, where $r_1, r_2, \cdots, r_n$ are pairwise different numbers and

$$\binom{r_i}{j} = r_i(r_i - 1) \cdots (r_i - j + 1)/j!.$$

Such determinants arise in seeking the general solution of an Euler linear differential equation

$$x^n y^{(n)} + A_1 x^{n-1} y^{(n-1)} + \cdots + A_{n-1} x y' + A_n y = 0$$

$(A_1, A_2, \cdots, A_n$ constants) in the form

$$y = C_1 x^{r_1} + C_2 x^{r_2} + \cdots + C_n x^{r_n}$$
$$(x > 0;\ C_1, C_2, \cdots, C_n, \text{ constants}).$$

The author shows that the determinant can be developed in terms of the Vandermond determinant of $r_1, r_2, \cdots, r_n$, the elementary symmetric functions of $r_1, r_2, \cdots, r_n$, and certain functions $\lambda_{p,q}$ (p, q integers) involving Stirling numbers. *L. M. Blumenthal* (Columbia, Mo.).

Čebotarev, G. N. On the problem of resolvents. Kazan. Gos. Univ. Uč. Zap. **114**, no. 2 (1954), 189–193. (Russian)

Developing further the ideas of A. Wiman [Nova Acta Soc. Sci. Upsal. (4) volumen extra ordinem editum **1927**], the author shows that the general algebraic equation of degree $n \geqq 21$ admits a resolvent with $n-6$ parameters.

E. R. Kolchin (Paris).

Morozov, V. V. On certain questions of the problem of resolvents. Kazan. Gos. Univ. Uč. Zap. **114**, no. 2 (1954), 173–187. (Russian)

The author's purpose is to clarify the resolvent problem, the relevant history of which runs: Hilbert, Math. Ann. **97** (1926), 243–250; A. Wiman, Nova Acta Soc. Sci. Upsal. (4) volumen extra ordinem editum **1927**; N. G. Čebotarev, Math. Ann. **104** (1931), 459–471; **105** (1931), 240–255; Izv. Fiz.-Mat. Obšč. Kazan. Univ. (3) **6** (1932–33), 5–22; Izv. Akad. Nauk SSSR Ser. Mat. **7** (1943), 123–146, [MR **6**, 113]. The author shows that a result due to N. G. Čebotarev, to the effect that the number of parameters of an equation with independently variable coefficients and alternating group can be reduced with the help of a certain irrationality by at most three, can not be strengthened. He then shows, by an example due to G. N. Čebotarev, that the proof of a result of N. G. Čebotarev on rational resolvents has a gap. The gap can be closed by certain supplementary hypotheses, which are satisfied in the case of an equation with independently variable coefficients and either symmetric or alternating group; the saved result is negative in this case, asserting that the number of parameters can be reduced by at most two. It is remarked that even for this case the problem without the rationality condition is far from solved, and various illuminating observations are made.

E. R. Kolchin (Paris).

Nečaev, V. I. Waring's problem for polynomials. Amer. Math. Soc. Transl. (2) **3** (1956), 39–89.

Translated from Trudy Mat. Inst. Steklov. **38** (1951), 190–243; MR **13**, 914.

Rinehart, R. F. The derivative of a matric function. Proc. Amer. Math. Soc. **7** (1956), 2–5.

The author investigates the question as to what extent and under what circumstances it is possible to replace in the usual definition $f'(A) = \lim h^{-1}(f(A+hI) - f(A))$ for a scalar complex valued function $f(z)$ the "incremental matrix" hI by a matrix H so that the limit is independent of the mode of approach of H to the zero matrix 0. Let $H \in S_\delta$, i.e. the set of all matrices H commuting with A for which $|h_{rs}| < \delta$ where $\delta > 0$. If $f(A+H) - f(A) = HQ$ for all $H \in S_\delta$ and if $\lim Q$ exists and is independent of the way H tends to 0, then $f^I(A) = \lim Q$ is shown to be equal to $f'(A) = (2\pi i)^{-1} \int_\sigma f(z)/(zI - A)^2 dz$, where σ is a set of simply closed curves, each of which encloses one root of A and within, and on, which $f(z)$ is analytic.

H. Schwerdtfeger (Melbourne).

See also: Allen, p. 1137; Fuhrmann, p. 1122; Palaj, p. 1123; Peres, p. 1137, Rogla, p. 1160; Stulenand Lehman, p. 1137.

Lattices

Hiraguti, Tosio. On the dimension of orders. Sci. Rep. Kanazawa Univ. **4** (1955), no. 1, 1–20.

This is largely a repetition of the author's previous work [Sci. Rep. Kanazawa Univ. **1** (1951), 77–94; **2** (1953), no. 1, 1–3; MR **17**, 19; **17**, 1045] with some extensions. Topics studied are: conditions for the existence of a linear order which is an extension of a given partial order; the dimension of a partial order, P, namely the cardinal number of the smallest set of linear orders whose intersection is P [Dushnik and Miller, Amer. J. Math. **63** (1941), 600–610; MR **3**, 73] conditions under which the dimension is unchanged, or changed by a limited amount, by deleting certain elements from the basic set; dimensions of sums and products of orders; relations between the dimension of an order and the cardinal

number of the set on which it is defined, both in general and in the special cases of small cardinal numbers.

P. M. Whitman (Silver Spring, Md.).

Curtis, H. J. A metrization problem concerning lattices. Proc. Amer. Math. Soc. 7 (1956), 319–330.

Wilcox has given conditions which insure that a metric topology for the space of points of an atomic semi-modular lattice may be extended to a Hausdorff topology of the entire lattice [Duke Math. J. 8 (1941), 273–285; MR 3, 56]. In the present paper the author finds additional conditions which insure that the extended topology, defined for the entire lattice, is also that of a metric space. In other words, he extends the metric and not merely the topology.

He assumes three new conditions in addition to Wilcox's original four conditions. One of the new conditions is that the original point space be separable. He then makes use of a metrization and imbedding theorem of Urysohn to extend the metric. It is not known to what extent the new conditions are necessary. *O. Frink.*

Jakubík, J. On the Jordan-Dedekind chain condition. Acta Sci. Math. Szeged 16 (1955), 266–269.

The author generalizes the example of G. Szasz of a distributive lattice in which not all maximal chains have the same cardinal number [same Acta 16 (1955), 89–91, 270; MR 17, 2]. The generalization is to the lattice of mappings from a class M onto the closed interval [0, 1]. The author proves that for any cardinal $\alpha \geqq c$ there exists a completely distributive lattice S_α with least element f_0 and greatest element f_1, and such that, for any cardinal β for which $c \leqq \beta \leqq \alpha$, there exists in S_α a maximal chain $R_\beta(f_0, f_1)$ of length β. *S. Gorn* (Philadelphia, Pa.).

Ville, J. Eléments de l'algèbre de Boole. Publ. Inst. Statist. Univ. Paris 4 (1955), 107–140.

Expository paper. *P. Halmos* (Chicago, Ill.).

Ellis, David; and Sprinkle, H. D. Topology of *B*-metric spaces. Compositio Math. 12 (1956), 250–262.

The paper is concerned with the Birkhoff-Kantorovich (B.-K.) sequential order topology in a σ-complete Boolean algebra B. It answers in the negative a question proposed in Problem 77 of Birkhoff's "Lattice theory" [Amer. Math. Soc. Colloq. Publ., v. 25, rev. ed., New York, 1948, p. 167; MR 10, 673], correcting an affirmative answer given by the first author [Bull. Amer. Math. Soc. 58 (1952), 662]. A B-metric space is formed by a set Σ and a mapping $d(x, y)$ of $\Sigma\Sigma$ into a σ-complete Boolean algebra such that $d(x, y)=0$ if and only if $x=y$, $d(x, y)=d(y, x)$, and $d(x, z) \subset d(x, y)+d(y, z)$. Putting d-$\lim x_i = x$ provided $\lim d(x, x_i)=0$ in B, it is seen that the B.-K. topology is a metric topology. It is erroneously stated that Σ is a Hausdorff space in its metric (and hence in its B.-K.) topology. A sequence $\{x_i\}$ of Σ is Cauchy provided $\lim_i \lim_j d(x_i, x_j)=0$, and B is called autometrized if $d(x, y)=xy'+x'y$ $(x, y \in B)$. It is shown that the autometrized B is complete. Other easily established properties of Cauchy sequences in Σ are proved. The paper concludes with an application to zero-dimensional spaces.

L. M. Blumenthal (Columbia, Mo.).

See also: Schützenberger, p. 1099.

Rings, Fields, Algebras

Faddeev, D. K. Simple algebras over a field of algebraic functions of one variable. Amer. Math. Soc. Transl. (2) 3 (1956), 15–38.

Translated from Trudy Mat. Inst. Steklov. 38 (1951), 321–344; MR 13, 905.

Chevalley, Claude. Sur les parties bornées d'un corps. J. Math. Pures Appl. (9) 35 (1956), 97–108.

Let $K \subset L$ be given fields. A subset B of L is said to be bounded if there exists a finite dimensional subspace M of the vector space L/K such that each element of B is expressible as the quotient of two elements of M. If B and B' are bounded subsets of L, then so are $1/B$, $B \cup B'$, $B \pm B'$, and BB'. Main result: If B is a bounded subset of L, then $K(B)/K$ is a field extension of finite type and if $u_1, \cdots, u_r \in B$ form any transcendence base for $K(B)/K$ then $[K(B):K(u_1, \cdots, u_r)]$ has an upper bound that depends only on B. The proof goes back to the notion of the degree of an algebraic variety in affine space.

M. Rosenlicht (Evanston, Ill.).

Cohn, Richard M. Specializations over difference fields. Pacific J. Math. 5 (1955), 887–905.

Introducing the language of specializations into the theory of difference fields, in analogy with algebraic geometry and with differential fields, the author considers the following question: given a difference field $\mathfrak{F}$ of characteristic 0 and elements $\beta_1, \cdots, \beta_q, \gamma_1, \cdots, \gamma_p$ of an extension of $\mathfrak{F}$, under what circumstances can a specialization $\bar{\beta}$ of $\beta=(\beta_1, \cdots, \beta_q)$ over $\mathfrak{F}$ be extended to a specialization of (β, γ)? In this connection he proves that if $\beta_1, \cdots, \beta_q$ are transformally independent over $\mathfrak{F}$ then there exists, independent of $\bar{\beta}$, a nonzero difference polynomial $D(\beta)$ over $\mathfrak{F}$, such that the desired specialization exists whenever $\mathfrak{F}\langle\bar{\beta}\rangle$ and $\mathfrak{F}\langle\beta, \gamma\rangle$ are compatible extensions of $\mathfrak{F}$ and $D(\bar{\beta}) \neq 0$. Counterexamples are given to show that the conditions of independence and of compatibility are not superfluous. *E. R. Kolchin.*

Nagahara, Takasi; and Tominaga, Hisao. On Galois theory of division rings. Proc. Japan Acad. 32 (1956), 153–156.

Nobusawa a récemment étendu la théorie de Galois des extensions infinies de corps commutatifs (Krull) à certains types de corps non commutatifs: la situation qu'il envisage est celle où on a un souscorps L d'un corps K, L étant corps d'invariants d'un groupe d'automorphismes de K, et où on suppose que pour tout élément $a \in K$, l'ensemble des transformés de a par le groupe des L-automorphismes de K est fini [Osaka Math. J. 7 (1955), 1–6; MR 16, 1084]. Les auteurs considèrent le cas où on suppose que pour toute partie finie S de K, le sous-corps de K engendré par L et la réunion des transformés de S par le group G des L-automorphismes de K, est de rang fini sur L (en tant qu'espace vectoriel à gauche). Ils énoncent sans démonstration toute une série de propositions, donnant notamment une condition pour que G soit localement compact, ainsi que des conditions permettant d'établir la correspondance du type usuel entre certains sous-groupes de G et certain corps intermédiaires entre L et K.

J. Dieudonné (Evanston, Ill.).

Dedecker, Paul. Une théorie algébrique des équations approchées. Bull. Soc. Math. France **83** (1955), 331–364.

The customary process of setting up approximate equations (in applied fields, e.g., meteorology) by neglecting terms of higher order of magnitude is studied here as an algebraic problem. The difficulty is that approximate equality is usually not additive nor subtractive, nor is it an equivalence relation [but it is multiplicative]. Order of magnitude is considered abstractly as a filtration of a ring A, i.e., a map f of the ring into a (totally) ordered semigroup M (with ∞ adjoined) such that (1) $f(a-b)\geqq \min(f(a), f(b))$, (2) $f(a\cdot b)\geqq f(a)\cdot f(b)$, (3) $f(0)=\infty$. One constructs the associated graded ring $\mathcal{A}$ of A (for $p \in M$ one has a direct summand $\mathcal{A}_p$, which is A_p/B_p, where $A_p=\{a:f(a)\geqq p\}$, $B_p=\{a:f(a)>p\}$), and a map $\varphi:A\to\mathcal{A}$, where $\varphi(a)$ is the element determined by a in $\mathcal{A}_p$, if $f(a)=p$ (and $\varphi(a)=0$, if $f(a)=\infty$). Any additive relation $a+b+\cdots+l=0$ in A defines certain relations in $\mathcal{A}$, constructed with the help of the f-values of $a, b, \cdots$; the first of these says that the sum of the φ-images of those of the $a, b, \cdots$, whose filtration is lowest possible, is 0; this corresponds to the usual step of setting up approximate equations. The author's point is that instead of approximate equations in the original situation one gets here exact equations in a new ("approximate") situation, namely in $\mathcal{A}$.

To establish contact with Analysis, the discussion is extended to rings with a derivation. Generalizations with M partially ordered are considered. Filtrations on the set of C^∞-functions are studied briefly. Relations with A. Weil's theory of local rings [Géométrie Différentielle, Colloq. Internat. Centre Nat. Rech. Sci., Strasbourg, 1953, pp. 111–117; MR **15**, 828] are considered.

H. Samelson (Ann Arbor, Mich.).

Kinohara, Akira. Theory of n-cocycles and n-cohomology groups in commutative rings. J. Sci. Hiroshima Univ. Ser. A. **19** (1955), 31–41.

The author is concerned with the normalization of symmetric cocycles for commutative rings and relativization with respect to subrings. This is done with a view to extending results of Kawada [Ann. of Math. (2) **54** (1951), 302–314; MR **13**, 324] on the relative 2-dimensional cohomology groups for valuation rings of p-adic number fields to arbitrary even dimensions. However, some of the results are manifestly false (in particular Theorem 2), and the "proofs" of others omit even the mere consideration of the critical questions, so that an appraisal of what is really secured in this paper cannot be given here.

G. P. Hochschild (Berkeley, Calif.).

Coimbra de Matos, A. Sur la notion de produit direct. Univ. Lisboa. Revista Fac. Ci. A. (2) **5** (1955–1956), 63–74.

An expository paper dealing with the elementary properties of the tensor product of modules over a ring. Special attention is paid to the case where the ring is commutative so that the tensor product inherits the module structure. *A. Rosenberg* (Princeton, N.J.).

Feller, Edmund H. The lattice of submodules of a module over a noncommutative ring. Trans. Amer. Math. Soc. **81** (1956), 342–357.

Let A, R be two associative rings with identity elements. A module M over the tensor product $A\otimes R$ is called an A-R module. M can be considered also both as an R-module and as an A-module. For an R-submodule N of M, the author considers subrings V of the centralizer $V^*(N)=\{a|a \in A,\ Na\subset N\}$. The V-radical of N is defined as the set $\{a|a \in V,\ Ma^t\subset N$ for some integer $t\}$. Let M satisfy the A.C.C. (ascending chain condition) for R-submodules, then the V-radical of an irreducible R-submodule N of M is a completely prime ideal in V. Thus, one can associate to every representation: $N=N_1\cap\cdots\cap N_k$ of N as an intersection of irreducible R-submodules N_i and to a fixed subring $F(N)\subset V_1^*(N_1)\cap\cdots\cap V^*(N_k)$, the set of the F-radicals $\mathfrak{p}_i$ of the modules N_i. The author now obtains decomposition and uniqueness theorems for A-R modules which satisfy the A.C.C. for R-submodules. This extends the Noether theory [Math. Ann. **83** (1921), 24–66] of the decompositions of ideals in commutative rings which satisfy the A.C.C. for ideals. In particular, one obtains a decomposition theory for right ideals by taking A and R to be respectively the ring of left and right multiplications of a given ring.

Next, the author considers completely indecomposable A-R modules which satisfy the A.C.C. and the D.C.C. for both R-submodules and A-submodules. For such modules M there is a one to one correspondence between the decomposition series of the R-submodules of M and of the right ideals of $A/\mathfrak{q}$, where $\mathfrak{q}$ is the anihilating ideal of M in A. Furthermore, the ring of A-endomorphisms of M is isomorphic with the ring $R/\mathfrak{q}'$, where $\mathfrak{q}'$ is the anihilating ideal of M in R. *S. A. Amitsur* (Jerusalem).

Rees, D. Valuations associated with ideals. Proc. London Math. Soc. (3) **6** (1956), 161–174.

Soient A un anneau d'intégrité noethérien et $\mathfrak{a}$ un idéal de A. Pour tout $x\neq 0$ dans A on note $v_\mathfrak{a}(x)$ le plus grand entier n tel que $x \in \mathfrak{a}^n$. Pour tout idéal $\mathfrak{b}$ de A notons $i(\mathfrak{b})$ l'idéal des éléments x de A entiers sur $\mathfrak{b}$ (c'est à dire vérifiant des équations de la forme $x^q+b_{q-1}x^{q-1}+\cdots+b_0=0$ avec $b_i \in \mathfrak{b}$). Supposons que la condition suivante soit vérifiée: (H) il existe un entier t tel que $i(\mathfrak{a}^{s+t+1})\cap\mathfrak{a}^s\subset\mathfrak{a}^{s+1}$ pour tout s suffisamment grand; alors il existe un nombre fini de valuations v_i du corps des fractions F de A admettant Z pour groupes des valeurs, et des entiers e_i et $t(\mathfrak{a})$ tels que $v_\mathfrak{a}(x)\leqq \min_i (e_i^{-1}v_i(x))\leqq v_\mathfrak{a}(x)+t(\mathfrak{a})$ pour tout $x\neq 0$ dans A. Dans la démonstration on se ramène, par utilisation d'un élément u superficiel d'ordre 1 pour $\mathfrak{a}$ et de l'anneau R réunion des $u^{-n}\mathfrak{a}^n$, au cas où $\mathfrak{a}$ est principal et où $i(\mathfrak{a})=\mathfrak{a}$. La condition (H) est vérifiée pour tout idéal $\mathfrak{a}$ de A lorsque, pour tout sous-anneau S de F de la forme $S=A[x_1, \cdots, x_q]$, la clôture intégrale de S est un S-module de type fini; ceci a lieu quand A est un anneau de coordonnées affines ou un anneau local géométrique. La condition (H) implique que le complété $\mathfrak{a}$-adique de A n'a pas d'éléments nilpotents. L'auteur étudie enfin les valuations discrètes v_i: leurs centres sur A contiennent $\mathfrak{a}$ et tout idéal premier isolé de $\mathfrak{a}$ figure parmi eux; lorsque $\mathfrak{a}^n$ est équidimensionnel pour tout n suffisamment grand, alors toute valuation v_i a pour centre un idéal premier isolé de $\mathfrak{a}$; enfin, si A est l'anneau de coordonnées affines d'une variété de dimension d, alors chaque v_i est de dimension $d-1$. *P. Samuel* (Clermont-Ferrand).

Mori, Shinziro. Über die eindeutige Darstellung der Ideale als Durchschnitt schwacher Primärideale. Proc. Japan Acad. **32** (1956), 83–85.

Soit R un anneau commutatif dans lequel tout idéal $\mathfrak{a}$ est intersection finie d'idéaux faiblement primaires. Pour que cette représentation soit unique pour tout idéal

$\mathfrak{a}$ de R, il faut et il suffit que, quels que soient les idéaux premiers $\mathfrak{p}$, $\mathfrak{p}'$ tels que $\mathfrak{p}\subset\mathfrak{p}'$ et l'idéal $\mathfrak{b}\subset\mathfrak{p}$, on ait $\mathfrak{p}'\mathfrak{b}=\mathfrak{b}$. Alors tout idéal $\mathfrak{a}$ de R n'a que des composantes primaires isolées. *P. Samuel* (Clermont-Ferrand).

Yoshida, Michio. On homogeneous ideals of graded Noetherian rings. J. Sci. Hiroshima Univ. Ser. A. **19** (1955), 43–49.

Soit A un anneau noethérien, gradué par un groupe totalement ordonné de degrés. Démonstration des résultats classiques sur la décomposition primaire des idéaux homogènes de A (méthode: associer à tout idéal $\mathfrak{a}$ de A le plus grand idéal homogène $\mathfrak{a}_H$ contenu dans $\mathfrak{a}$). Le théorème de Krull sur les chaines d'idéaux premiers est valable si on se limite aux idéaux premiers homogènes. De même les résultats sur les longueurs d'idéaux primaires. Une remarque donne une démonstration simplifiée du „Primidealsatz" de Krull. *P. Samuel.*

Moisil, Gr. C. Une définition des nombres idéaux. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 831–841. (Romanian. Russian and French summaries)

A necessary and sufficient condition that the integers a, b should have a greatest common divisor $d>1$ is that for some integer $x\not\equiv 0 \pmod a$, $xb\equiv 0 \pmod a$. Also, if and only if for every integer x, $xb\equiv 0 \pmod a$ implies $xc\equiv 0$ $\pmod a$, then $(a,b)|c$. The extension of these results to integral domains is used as a definition of ideal numbers. In order to do this, the author defines two relations among the elements of the integral domain $\mathfrak{o}$: (i) $F(\omega_0, \cdots, \omega_r)$ is said to hold among the elements $\omega_i \in \mathfrak{o}$ $(i=0, \cdots, r)$, if and only if there exists some element $\lambda_0 \in \mathfrak{o}$, such that $\lambda_0\not\equiv 0 \pmod{\omega_0}$, $\lambda_0\omega_i\equiv 0 \pmod{\omega_0}$ for $i=1, 2, \cdots, r$; and (ii) $D(\omega_0, \cdots, \omega_r|\theta)$ is said to hold if and only if, for every $\lambda \in \mathfrak{o}$, $\lambda\omega_i\equiv 0 \pmod{\omega_0}$ $(i=1, 2, \cdots, r)$ implies $\lambda\theta\equiv 0$ $\pmod{\omega_0}$. Both relations are symmetric in the ω_i's $(i=0, 1, \cdots, r)$. Two sets are said to be associate, $\{\omega_0, \cdots, \omega_r\}\sim\{\theta_0, \cdots, \theta_s\}$ provided that $D(\omega_0, \cdots, \omega_r|\theta_i)$ $(i=0, 1, \cdots, s)$ and $D(\theta_0, \cdots, \theta_s|\omega_j)$ $(j=0, 1, \cdots, r)$. This is an equals relation; the corresponding equivalence classes of finite subsets of $\mathfrak{o}$ are called ideal numbers. Under this definition, the class of elements $\{\omega_0, \cdots, \omega_r\}$ such that $F(\omega_0, \cdots, \omega_r)$ does not hold corresponds to the unit ideal (1). With these definitions the usual theorems in the elementary theory of ideals may be proven.

E. Grosswald (Philadelphia, Pa.).

Krull, Wolfgang. Eine Bemerkung über primäre Integritätsbereiche. Math. Ann. **130** (1956), 394–398.

Let A be a domain of integrity, K its field of quotients; A is a primary ring if the set of non-units in A is the unique prime ideal $\mathfrak{p}$ of A. A valuation ring corresponding to a real-valued valuation is primary and in addition is completely integrally closed (i.e. if $x \in K$ is such that there exists an $a \in A$ such that $a\neq 0$ and $ax^n \in A$ for all integers $n\geqq 1$, then $x \in A$). The converse does not hold, and the question arises to find additional conditions characterizing such a valuation ring (also called maximal ring, from a well-known property proved by the author). A first condition even ensures that A will be a discrete valuation ring: this occurs when for n large enough $\mathfrak{p}^n$ is contained in a finitely generated ideal. Suppose this condition is not satisfied (in which case A is said to be an infinite primary ring); then the author obtains the following criterion. For any pair of finitely generated ideals $\mathfrak{e}$, $\mathfrak{e}_1$, such that $\mathfrak{e}\subset\mathfrak{e}_1$, let $g(\mathfrak{e}, \mathfrak{e}_1)$ be the largest integer m such that $\mathfrak{e}_1{}^m\supset\mathfrak{e}$ and $\mathfrak{e}_1{}^m\neq\mathfrak{e}$, and $k(\mathfrak{e}, \mathfrak{e}_1)$ the smallest integer n such that $\mathfrak{e}_1{}^n\subset\mathfrak{e}$; let

$$d(\mathfrak{e}, \mathfrak{e}_1)=(k(\mathfrak{e}, \mathfrak{e}_1)-g(\mathfrak{e}, \mathfrak{e}_1))/g(\mathfrak{e}, \mathfrak{e}_1).$$

The criterion is then that for any pair of finitely generated ideals $\mathfrak{e}$, $\mathfrak{e}'$, there should exist a finitely generated ideal $\mathfrak{e}_1$ containing $\mathfrak{e}$ and $\mathfrak{e}'$ and such that $d(\mathfrak{e}, \mathfrak{e}_1)$ and $d(\mathfrak{e}', \mathfrak{e}_1)$ are arbitrarily small. *J. Dieudonné* (Evanston, Ill.).

Ribenboim, P. Un théorème sur les anneaux primaires et complètement intégralement clos. Math. Ann. **130** (1956), 399–404.

The notations and terminology being those of the preceding review, the author gives another criterion for a primary, completely integrally closed ring to be a maximal ring. Let $t \in \mathfrak{p}$, $t\neq 0$, $a \in K$; let $\mu_t(a)$ be the supremum of the set of rational numbers m/n such that $Aa^n\subset At^m$, and $\nu_t(a)$ the infimum of the set of rational numbers m/n such that $At^m\subset Aa^n$; if $\mu_t(a)=\nu_t(a)=s$, the ideal Aa is called the sth power of At, and one writes $Aa=(At)^s$. The condition is then that $\mathfrak{p}$ should be the union of an ascending sequence of principal ideals At_r $(r\geqq 1)$ such that: 1) for $r>1$, $At_{r-1}=(At_r)^{s_r}$ for a suitable real s_r; 2) if a, b in A are such that $a \notin At_r$, $b \notin At_r$, then $ab \notin At_{r-1}$.

J. Dieudonné (Evanston, Ill.).

Drazin, M. P. Engel rings and a result of Herstein and Kaplansky. Amer. J. Math. **77** (1955), 895–913.

The author is concerned with generalizing a result due to Kaplansky [Canad. J. Math. **3** (1951), 290–292; MR **13**, 101] and the reviewer [ibid. **5** (1953), 238–241; MR **14**, 719]. To do this he introduces the idea of a K-ring, whose explicit definition we need not give here. Unfortunately the proof of the key result, Theorem 2.1, is incomplete, and its completion would require something like a verification of the conjecture due to Koethe that a ring possessing a non-trivial nil right ideal possesses a non-trivial nil two-sided ideal. Theorem 2.1 is needed either directly or indirectly in the proof of almost all the remaining results, so it is not easy to assess what results can be salvaged in face of the serious lacuna in the proof of Theorem 2.1. *I. N. Herstein* (Philadelphia, Pa.).

Leavitt, William G. Modules over rings of words. Proc. Amer. Math. Soc. **7** (1956), 188–193.

This paper is essentially a continuation of the author's paper [An. Acad. Brasil. Ci. **27** (1955), 241–250; MR **17**, 578]. By considering a ring of words, it is shown that there exists a ring K (always with unity and without divisors of zero) such that every free K-module has invariant basis number and contains infinite sets of independent elements. An example is given of a ring K with this property such that K is not imbeddable in a division ring. On the other hand, an example is given of a ring K such that a free K-module has invariant basis number if and only if it has a basis of length 1. *R. E. Johnson* (Northampton, Mass.).

Berman, S. D. On the equation $x^m=1$ in an integral group ring. Ukrain. Mat. Ž. **7** (1955), 253–261. (Russian)

Detailed exposition of results previously announced in Dokl. Akad. Nauk SSSR (N.S.) **91** (1953), 7–9 [MR **15**, 99]. *E. R. Kolchin* (Paris).

Hewitt, Edwin; and Zuckerman, Herbert S. Finite dimensional convolution algebras. Acta Math. **93** (1955), 67–119.

Let G be an arbitrary semigroup, i.e., a system pos-

sessing an associative multiplication. Let $\mathfrak{F}$ be a vector subspace of the complex vector space $\mathfrak{F}_1(G)$ of all complex-valued functions on G, such that the function ${}_x f$ defined by ${}_x f(y)=f(xy)$ belongs to $\mathfrak{F}$ whenever f does. Let $\mathfrak{L}$ be a vector subspace of the algebraic dual of $\mathfrak{F}$, and suppose that (i) the function g defined on G by $g(x)=L_y(f(xy))$ belongs to $\mathfrak{F}$ whenever $f \in \mathfrak{F}$, $L \in \mathfrak{L}$, and (ii) the functional N defined on $\mathfrak{F}$ by $N(f)=M_x(L_y(f(xy)))$ belongs to $\mathfrak{L}$ whenever $f \in \mathfrak{F}$, $L \in \mathfrak{L}$ and $M \in \mathfrak{L}$. Then $\mathfrak{L}$ is said to be a convolution algebra, and $N=M*L$ is called the convolution of M and L. The operation of convolution is then associative.

The authors show that the complex group algebra of a finite group, the algebra of formal power series with complex coefficients, and the $\mathfrak{L}_1$ algebra of a locally compact group are all convolution algebras. Again, every finite-dimensional algebra over the complex field is a convolution algebra. Other examples are also given.

The most important case for the present paper is that in which G is a finite semigroup, $\mathfrak{F}=\mathfrak{F}_1(G)$, and $\mathfrak{L}=\mathfrak{L}_1(G)$ is the algebraic dual of $\mathfrak{F}_1(G)$; $\mathfrak{L}_1(G)$ is called the $\mathfrak{L}_1$ algebra of G, and is the natural generalization of the group algebra of a finite group. $\mathfrak{L}_1(G)$ is commutative if and only if G is commutative; a finite-dimensional algebra over the complex field is isomorphic with the $\mathfrak{L}_1$ algebra of some finite semigroup if and only if it possesses a basis that is closed under multiplication.

§ 2 of the paper is devoted to results on semigroups. First there is a discussion of the operation of adjoining various types of elements, such as a zero or a unit, to a given semigroup. An element x is then defined to be of finite order if there exist integers k and l such that $k \geqq l \geqq 1$ and $x^{k+l}=x^l$. If r is the smallest integer such that $x^r=x^s$ for some $s<r$, we write $l_x=s$, $k_x=r-s$. The following structure theorems are then proved. (i) A commutative semigroup in which every element is idempotent can be represented as a system of subsets of itself, multiplication being represented by intersection. (ii) A commutative semigroup in which every element is of finite order and $l_x=1$ for all x consists of a set of disjoint groups. (iii) A commutative semigroup in which every element is of finite order can be partitioned into a set of disjoint semigroups T_a such that, if G^0 is the sub-semigroup $\{x:l_x=1\}$, $T_a \cap G^0$ is a group. (iv) A semigroup G in which every element is of finite order and the left cancellation law holds can be written as a direct product $G=J\times H$, where J is a group and H is a semigroup with the law $xy=y$. (v) A semigroup in which every element is idempotent consists of a set of disjoint semigroups, each of which is isomorphic to a set of pairs (y, z), with the law

$$(y_1, z_1)(y_2, z_2)=(y_1, z_2).$$

If G is a semigroup, a semicharacter of G is a complex function χ on G such that $\chi(xy)=\chi(x)\chi(y)$ $(x, y \in G)$, and $\chi(x)\neq 0$ for some $x \in G$. If G is finite, and there is a semicharacter χ such that $\chi(x)\neq 0$ for all x and $\chi(x)\neq\chi(y)$ when $x\neq y$, then G is a commutative group. More generally, if every element of G is of finite order, then G is commutative and $l_x=1$ for all x if and only if, for some semicharacter χ, $\chi(x)\neq\chi(y)$ $(x\neq y)$. The effect of adjunction operations on the semicharacters is described. Conditions are given for the set $\bar{G}$ of all semicharacters to be itself a semigroup, and for $\bar{\bar{G}}$ to be isomorphic with G or with G^0.

The remainder of the paper is devoted to the $\mathfrak{L}_1$ algebras of finite semigroups. It is shown that if the semigroup G is of order n, then $\mathfrak{L}_1(G)$ has a faithful matrix representation of order at most $n+1$. A finite-dimensional semisimple algebra is isomorphic to the $\mathfrak{L}_1$ algebra of some semigroup if and only if its representation as a direct sum of full matrix algebras contains a one-dimensional summand. The $\mathfrak{L}_1$ algebras of a number of special types of semigroups are constructed, and the effect of adjunction operations on the $\mathfrak{L}_1$ algebra of a semigroup is discussed. A few special cases in which a semigroup is determined by its $\mathfrak{L}_1$ algebra are indicated.

The last part of the paper is concerned with the duality existing between ideals in $\mathfrak{L}_1(G)$ and invariant subspaces of $\mathfrak{F}_1(G)$. It is shown that correspondence can be set up between the homomorphisms of $\mathfrak{L}_1(G)$ onto the complex field and the semicharacters of G, and between the homomorphisms of $\mathfrak{L}_1(G)$ onto the one-dimensional algebra Z_1 (with zero products) and the functions belonging to the subspace

$$\mathfrak{T}(G)=\{f:f(xy)=0\ (x, y \in G)\} \text{ of } \mathfrak{F}_1(G).$$

The intersecting minimal invariant subspaces $\mathfrak{S}$ of $\mathfrak{F}_1(G)$ are those for which $\mathfrak{S}\cap\mathfrak{T}(G)=\{0\}$; there is a finite linearly independent set of these, each of which is spanned by the coefficients of an irreducible matrix representation of G; their direct sum intersects $\mathfrak{T}(G)$ in $\{0\}$, and has as its annihilator the radical of $\mathfrak{L}_1(G)$. The structure of the radical is completely determined in the commutative case.

Appendices to the paper contain lists of all semigroups of orders 2 and 3, and of all those of orders 5 and 6 having semisimple $\mathfrak{L}_1$ algebras.

Many other results, too numerous to mention here, are to be found in the paper. *F. Smithies.*

Wooyenaka, Yuki. On Newman algebra. III. Proc. Japan Acad. **31** (1955), 66–69.

In parts I and II of this series [same Proc. **30** (1954), 170–175, 562–565; MR **16**, 329] postulate sets I^* and II^* were given for a Newman algebra (= direct sum of a Boolean lattice and Boolean associative ring with unity). It is now shown that 6) $a(b+c)=ab+ac$ of I^* and II^* and either 3) $a+b=b+a$ of I^* or 8′) $a+b'b=b'b+a$ of II^* can be replaced by 6′) $a(b+c)=ca+ba$. Examples establish independence for the new axioms III^*. *W. Givens.*

Almeida Costa, A. Über die nichtassoziativen Ringe, die halbeinfache Moduln sind. Univ. Lisboa. Revista Fac. Ci. A. (2) **5** (1955–1956), 75–102.

Let $\mathfrak{X}$ be a non-associative ring, $\mathfrak{A}$ the ring of endomorphisms of the additive group of $\mathfrak{X}$, Ω a set of operators for $\mathfrak{X}$, and Ω_r the (associative) subring of $\mathfrak{A}$ generated by the endomorphisms corresponding to the elements of Ω. Let $\mathfrak{X}_l$ and $\mathfrak{X}_r$ be the subrings of $\mathfrak{A}$ generated by the left and right multiplications, respectively, of $\mathfrak{X}$. The transformation ring $\mathfrak{T}$ of $\mathfrak{A}$ is defined as the subring of $\mathfrak{A}$ generated by Ω_r, $\mathfrak{X}_l$, $\mathfrak{X}_r$, and the identity endomorphism. The author studies the relationship between the associative ring $\mathfrak{T}$ and certain of its subrings, and the given non-associative ring $\mathfrak{X}$. Particular attention is paid to those rings $\mathfrak{X}$ whose additive groups are completely reducible, homogeneous Ω_r-modules, so that the theory is a generalization of the theory of non-associative algebras over a field. Most of the results in the paper are direct extensions of the corresponding results for finite dimensional non-associative algebras, which are due to Albert [Ann. of Math. (2) **43** (1942), 685–707; MR **4**, 186]. There is also some overlapping with part II of a paper by Jacobson [Trans. Amer. Math. Soc. **57** (1945), 228–245; MR **6**, 200]. *C. W. Curtis* (Los Angeles, Calif.).

Witt, Ernst. Die Unterringe der freien Lieschen Ringe. Math. Z. **64** (1956), 195–216.

This paper contains a discussion of many questions centering around the problem of constructing a set of generators and relations for a subring $\mathfrak{U}$ of a Lie ring $\mathfrak{L}$ from a knowledge of a set of generators and relations for $\mathfrak{L}$. In particular it is proved that every subalgebra of a free Lie algebra with coefficients in a field is free. This theorem had been announced previously by the author [Math. Z. **58** (1953), 113–114; MR **15**, 5], and has been proved independently by Širšov [Mat. Sb. N.S. **33(75)** (1953), 441–452; MR **15**, 596].

Let $\mathfrak{L}$ be a Lie ring with bracket operation $x\bigcirc y$, and with operators in a commutative ring Γ with an identity element. Let X be a set of generators of $\mathfrak{L}$, and let $x\to|x|>1$ be a fixed mapping of X into the semigroup $\mathfrak{S}$ of elements >1 in a commutative ordered group written multiplicatively. For each $s\in\mathfrak{S}$, let $\mathfrak{L}_s$ be the Γ-module spanned by all possible Lie products $x_1\bigcirc\cdots\bigcirc x_n$, $|x_1|\cdots|x_n|\leqq s$. Let $\mathfrak{U}$ be a subring of $\mathfrak{L}$. As a first hypothesis, assume (I) that $\mathfrak{L}$ has a basis B over Γ which contains a basis for $\mathfrak{U}$ and a basis for each $\mathfrak{L}_s$. Let $\tilde{\mathfrak{L}}$ and $\tilde{\mathfrak{U}}$ be the Birkhoff-Witt enveloping rings of $\mathfrak{L}$ and $\mathfrak{U}$ respectively; then $\tilde{\mathfrak{U}}$ may be viewed as a subring of $\tilde{\mathfrak{L}}$ which contains the identity element of $\tilde{\mathfrak{L}}$. For each $s\in\mathfrak{S}$, let X_s be the Γ-submodule of $\tilde{\mathfrak{L}}$ generated by all products $x_1\cdots x_n$, $n\geqq0$, $x_1\in X$, $|x_1|\cdots|x_n|\leqq s$. Then $\mathfrak{L}\cap X_s=\mathfrak{L}_s$. Now let $\mathfrak{U}_s=\mathfrak{U}\cap X_s$, $\mathfrak{B}_s=\mathfrak{U}\cap\sum_{t_i<s,\ \Pi t_i\leqq s}\mathfrak{U}_{t_1}\cdots\mathfrak{U}_{t_n}$, and assume (II): for each $s\in\mathfrak{S}$, the module $\mathfrak{U}_s/\mathfrak{B}_s$ has a basis over Γ. Let $\{y_s\}$ be a set of representatives in $\mathfrak{U}_s$ of the cosets which form a basis of $\mathfrak{U}_s/\mathfrak{B}_s$ over Γ, and let Y be the totality of all $\{y_s\}$, $s\in\mathfrak{S}$. Then Y is a set of generators of $\mathfrak{U}$. The author next introduces the family $\{c_k\}$ of ordered basis elements of $\tilde{\mathfrak{L}}$ which are standard monomials in those elements of B which do not belong to $\mathfrak{U}$, and defines a representation $a\to(d_{ik}(a))$ of $\tilde{\mathfrak{L}}$ by column finite matrices with coefficients in $\tilde{\mathfrak{U}}$ by means of the relations $ac_k=\sum c_i d_{ik}(a)$, i, $k\geqq0$ $(c_0=1)$, $d_{ik}(a)\in\tilde{\mathfrak{U}}$. To each non-commutative polynomial $r(x)$ in the generators X of $\mathfrak{L}$ correspond certain polynomials $d_{i0}(r)$ in the generators Y of $\mathfrak{U}$. The main theorem can now be stated as follows. If $\{r(x)=0\}$ is the totality of relations among the elements of X, then the relations $\{d_{i0}(r)=0\}$ among the generators Y of $\mathfrak{U}$ generate all associative relations among the elements of Y. In particular, if X is a set of free generators of $\mathfrak{L}$, then Y is a set of free generators of $\mathfrak{U}$. Because hypotheses (I) and (II) are automatically satisfied if Γ is a field, this theorem implies that every subalgebra of a free Lie algebra over Γ is free.

Now let Γ be a principal ideal domain. A second major result states that under a further hypothesis, every homogeneous subring $\mathfrak{U}$ of a free Lie ring $\mathfrak{L}$ such that the factor module $\mathfrak{L}/\mathfrak{U}$ is torsion-free possesses a set of free generators which are homogeneous. A construction is given for a certain minimal basis of the free Lie ring; another construction has been given by Marshall Hall [Proc. Amer. Math. Soc. **1** (1950); 575–581, MR **12**, 388]. The methods developed by the author are also applied to restricted Lie rings [see Jacobson, Trans. Amer. Math. Soc. **50** (1941), 15–25; MR **3**, 103]. *C. W. Curtis.*

Kokoris, L. A. On a class of almost alternative algebras. Canad. J. Math. 8 (1956), 250–255.

Let γ, δ be elements of a field F satisfying $\gamma^2-\delta^2+\delta=1$. Albert calls an algebra A over F an algebra of type (γ, δ) [Portugal. Math. **8** (1949), 23–36; MR **11**, 316] in case the identities $(z, x, y)+\gamma(x, z, y)+\delta(y, z, x)=0$, $(x, y, z)=\gamma(x, z, y)+(\delta-1)(y, z, x)$ are satisfied, where (x, y, z) is the associator $(xy)z-x(yz)$. In this paper the structure of these algebras with $\delta\neq0$, 1 and characteristic $\neq2$, 3, 5 is investigated.

These algebras are power-associative. There is a Peirce decomposition relative to any idempotent e which has the multiplicative properties of the decomposition for associative algebras. If the radical is defined to be the maximal nilideal, then any semisimple algebra is a direct sum of simple ones. Each simple algebra either is associative or has a unity element which is an absolutely primitive idempotent.

R. D. Schafer (Storrs, Conn.).

See also: Deskins, p. 1052; Lombardo-Rodice, p. 1122; Northcott, p. 1134; Ree and Wisner, p. 1051; Seligman, p. 1108; Collected works of George Miller, p. 1037.

Groups, Generalized Groups

Benado, Mihail. Über die Zerlegungen einer Gruppe in direkten Produktfaktoren. I. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 241–248. (Romanian. Russian and German summaries)

Benado, Mihail. Über die Zerlegungen einer Gruppe in direkten Produktfaktoren. II. Acad. Repub. Pop. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. **7** (1955), 249–254. (Romanian. Russian and German summaries)

In I, the author considers pairs of operator endomorphisms ω_1 and ω_2 on an operator group G such that $\omega_1+\omega_2=1$. In order that G split into $G_1\oplus G_2$ with ω_i an operator automorphism on G_i it is necessary and sufficient that the operator center (the largest operator subgroup Z in the center) so split. If G has this splitting property for all such pairs of endomorphisms, then so does each direct summand of G; while if the operator center of G has this splitting property, then so does G. Various modifications of this splitting property are treated with similar results. In order that the factors of one direct decomposition of G can be introduced (by one-to-one exchange) into a second direct decomposition it is necessary and sufficient that corresponding exchanges take place in Z and that essentially identical decompositions are induced in G/Z. I suffers from numerous minor misprints and has uninformative summaries. In II, it is stated that two direct decompositions of the universal supremum of a complete modular lattice [complete modular in the sense of Kuroš, The theory of groups, 2nd ed., Gostehizdat, Moscow, 1953, p. 288; MR **15**, 501] have a common refinement if and only if for each pair of projection endomorphisms, one from the first decomposition, the other from the second, the members of the pair commute on the universal supremum. [Cf. ibid. p. 103.] If two direct decompositions of G induce decompositions of Z which have a common refinement, then the two direct decompositions of G each have refinements which are isomorphic with certain "exchange" properties. No proofs are given. For English readers, pertinent references are to Baer [Trans. Amer. Math. Soc. **61** (1947), 508–516; **62** (1947), 62–98; **64** (1948), 519–551; Bull. Amer. Math. Soc. **54** (1948), 167–174; MR **8**, 563; **9**, 134; **10**, 425; **9**, 410].

F. Haimo **(St. Louis, Mo.).**

★**Higman, Bryan. Applied group-theoretic and matrix methods.** Oxford, at the Clarendon Press, 1955. xiii+454 pp. $9.60.

This very commendable book gives a fairly far-going introduction to the theory of groups and group representations, and a great variety of applications to physics and chemistry. In principle very little mathematical knowledge is presupposed from the reader. All basic notions for groups, vector spaces and matrix algebras are carefully introduced and are illustrated by simple examples. Still in many respects the subject matter covered is quite extensive and reaches problems far from elementary. The style is at complete variance with what has become the conventional style of mathematical textbooks. Little attention is paid to strict rigor in the presentation, and the reader will often find, even in the course of reasoning, hints or analogies designed to support his intuitive understanding. By these features the book is certainly well adapted to students in physics with little mathematical background but some familiarity with the spirit of theoretical physics.

In Part I, the theory of finite groups and the matrix algebra are exposed and used to construct the representation theory of finite groups. A description is given of all finite groups of order up to 24.

Part II extensively studies the application of finite groups to crystals and molecules. It also deals with factor analysis, a matter usually classified as belonging to mathematical statistics but here illustrated by means of a chemical application.

Continuous groups are considered not too thoroughly in Part III, with strong reliance on the analogies with finite groups. The linear groups are studied on the basis of an analysis of the symmetric groups, here handled more extensively than in Part I. A chapter is devoted to tensors, with special attention to their application to crystals. The four last chapters are brief introductions to four fields of theoretical physics where groups play a considerable role: relativity (with a shade of general relativity), quantum mechanics, the quantum theory of molecules and the "Fundamental theory" [Cambridge, 1946; MR **11**, 144] of Eddington, a slightly unexpected but not uninteresting selection, to be undoubtedly explained by the author's admiration for Eddington's attempt.

The bibliography is limited to a fairly small number of classical textbooks, ordered by subject. *L. Van Hove.*

Struik, Ruth Rebekka. On associative products of groups. Trans. Amer. Math. Soc. **81** (1956), 425–452.

The paper contains some remarkable contributions to the theory of group-theoretical constructions. It is known that the free and the direct product of groups as algebraic operations satisfy the following properties: (a) commutativity; (b) associativity; (c) the product contains subgroups which generate the product; (d) these subgroups are isomorphic to the original groups; (e) the intersection of a given one of these subgroups with the normal subgroups generated by the rest of these subgroups is the identity; (f) (MacLane's postulate) if each of these subgroups is replaced by a factor group of the same subgroup, then the resulting group is the same as the product of the factor groups. A regular resp. fully regular product on groups, introduced by O. N. Golovin [Mat. Sb. N.S. 27(69) (1950), 427–454; MR **12**, 672], is a product which satisfies conditions (c) and (e) resp. (a) through (e). Golovin showed that a regular product necessarily satisfies (d) and constructed a denumerable number of fully regular products which he called nilpotent products.

In the present paper the regular products are systematically discussed. Answering a problem of Golovin, it is shown that there exist nonassociative regular products, even commutative nonassociative regular products. Moreover, a denumerable number of commutative, regular products is defined and proved associative. These products are special cases of verbal products of S. Moran [Thesis, London, 1954] and contain as special cases Golovin's nilpotent products. Finally property (f) is investigated and it is shown that all the commutative, regular products mentioned so far satisfy this postulate; however, there exist fully regular products which do not satisfy (f), i.e. MacLane's postulate is independent of the other postulates. *A. Kertész* (Debrecen).

Ree, Rimhak; and Wisner, Robert J. A note on torsion-free nil groups. Proc. Amer. Math. Soc. **7** (1956), 6–8.

The investigations of the authors join with two of Szele's papers [Math. Ann. **121** (1949), 242–246; Math. Z. **54** (1951), 168–180; MR **11**, 496; **13**, 316]. An additive abelian group G is a nil group or strongly nil group respectively, if there is no non-trivial associative or non-associative ring, which has G as its additive group. If K, L, M are subgroups of the additive rationals R^+ which contain the rational integers, (K, L, M) denotes the set of all rationals t for which $tKLCM$. Let G be the (discrete) direct sum of the (integer containing) subgroups H_λ, $\lambda \in \Lambda$ of R^+. (1) G is strongly nil if and only if $(H_\lambda, H_\mu, H_\nu)=0$ for every λ, μ, ν in Λ. (2) G is nil if and only if $(H_\lambda, H_\mu, H_\nu)=0$ for every λ, μ, ν in Λ satisfying either (i) $\lambda=\mu=\nu$ or (ii) $\lambda\neq\nu$, $\mu\neq\nu$. Moreover, the conditions of (2) imply those of (1), i.e. for groups which are direct sums of subgroups of R^+, the notations "nil" and "strongly nil" are equivalent. A lemma gives the condition $(H_\lambda, H_\mu, H_\nu)=0$ in an explicit form. *A. Kertész.*

Feit, W. On a conjecture of Frobenius. Proc. Amer. Math. Soc. **7** (1956), 177–187.

Let $\mathfrak{G}$ be a group of order mq, $(m, q)=1$. Suppose that there exists a subgroup $\mathfrak{Q}$ of order q such that (1) any element of $\mathfrak{G}$ of order dividing q is conjugate to some element of $\mathfrak{Q}$. Each element $G \in \mathfrak{G}$ can be written uniquely as a product $G=QM=MQ$, where $M^m=Q^q=1$. If ϱ is an irreducible character of $\mathfrak{Q}$, $\chi(G)=\varrho(Q_1)$, where Q_1 is some element of $\mathfrak{Q}$ conjugate to Q, defines a single valued function χ on $\mathfrak{G}$. Under the additional assumption that (2) for $Q \in \mathfrak{Q}$ the number of elements of the centralizer $\mathfrak{C}(Q)$ of Q whose orders divide m is the q-factor of the order of $\mathfrak{C}(Q)$, the author proves that χ is an irreducible character of $\mathfrak{G}$ by applying a theorem of R. Brauer [Ann. of Math. (2) **57** (1953), 357–377; p. 357; MR **14**, 844]. He concludes that under the hypotheses (1) and (2) the elements of $\mathfrak{G}$ whose order divide m constitute a normal subgroup of $\mathfrak{G}$ [of order m]. From this result he derives the theorem obtained by relacing the hypothesis that $\mathfrak{Q}$ be equal to its normalizer, by the hypothesis that $\mathfrak{G}$ contain exactly m elements whose orders divide m, in the theorem of Frobenius [A. Speiser, Theorie der Gruppen endlicher Ordnung, 3rd ed., Springer, Berlin, 1937, pp. 202–3]; actually a more general result is obtained. If $\mathfrak{G}$ of order mq, $(m, q)=1$, contains a normal subgroup of order m, then there exists a subgroup of order q by Schur's theorem, and the author proves that (1) and (2) are satisfied.

D. G. Higman (Missoula, Mont.).

Hall, P. Theorems like Sylow's. Proc. London Math. Soc. (3) **6** (1956), 286–304.

The subgroup H of the finite group G is termed an S_ω-subgroup of G, for ω a set of primes, if every prime divisor of the order of H belongs to ω whereas none of the prime divisors of the index $[G:H]$ belongs to ω. A finite group G may or may not have one of the following three properties which form the center of the present investigation: (E_ω) G possesses at least one S_ω-subgroup. (C_ω) G satisfies E_ω and S_ω-subgroups of G are conjugate in G. (D_ω) G satisfies C_ω and every ω-subgroup of G is part of an S_ω-subgroup of G. Of the many important results arising from this investigation we mention only the following ones: (1) If the normal subgroup K of G possesses a nilpotent S_ω-subgroup, if G/K satisfies D_ω and if every ω-subgroup of G/K is soluble, then D_ω is satisfied by G and every ω-subgroup of G is soluble. (2) If every composition factor of G is either an ω-group or of order prime to every prime in ω, and if L is a soluble ω-subgroup of G, H an S_ω-subgroup of G, then L is conjugate to a subgroup of H. (3) If ω_0, ω_1 and ω_2 are mutually exclusive sets of primes, and if every composition factor of G is either an $\omega_0\omega_1$-group or an $\omega_0\omega_2$-group, then $G=HK$ is the product of an $\omega_0\omega_1$-subgroup H and an $\omega_0\omega_2$-subgroup K whose intersection $H\cap K$ is a soluble ω_0-group. *R. Baer.*

Piccard, Sophie. Un problème de structure des groupes d'ordre fini. J. Math. Pures Appl. (9) **35** (1956), 141–160.

Let a group G of finite order N be generated by m generators A_j, connected by functional relations $f_i(a_1\cdots a_m)=1$, such that the sum of the exponents on a certain subset of the generators a_j is divisible by n in each f_i. Then G is said to enjoy the property P (mod n) with respect to that subset. Should the property hold for each of t elements, or more generally, for each of t subsets, then the elements of G for which in any representation as a product of powers of generators the corresponding exponent sums are congruent to $(i_1, i_2, \cdots, i_t)$ (mod n) form a coset $M_{i_1\cdots i_t}$ of the normal subgroup $M_{0\cdots 0}$. The abelian factor group Γ of order n^t (called p^t by mistake in Proposition 3) is the direct product of t cyclic groups of order n. The factor group Γ is generated by t basis elements, corresponding to a set of independent cosets of $M_{0\cdots 0}$. Thus a group G enjoying the property P for t basis elements cannot be generated by less than t elements, and such basis elements must be taken independent cosets of $M_{0\cdots 0}$.

J. S. Frame (East Lansing, Mich.).

Deskins, W. E. Finite Abelian groups with isomorphic group algebras. Duke Math. J. **23** (1956), 35–40.

First, the correspondence is studied from a subgroup $\mathfrak{H}$ of the finite group $\mathfrak{G}$ to the left ideal, $\mathfrak{L}(\mathfrak{H})$, in the group algebra $\mathfrak{A}$ of $\mathfrak{G}$ over a field $\mathfrak{F}$, generated by all $H-1$ for H in $\mathfrak{H}$. The following extension of a result of Jennings [Trans. Amer. Math. Soc. **50** (1941), 175–185; MR **3**, 34] is obtained: if $\mathfrak{F}$ has characteristic p, H order p^e, then $\mathfrak{J}$, the intersection of $\mathfrak{L}(\mathfrak{H}')$ for all $\mathfrak{H}'$ conjugate to $\mathfrak{H}$, is a nilpotent ideal in $\mathfrak{A}$. Second, for $\mathfrak{F}$ of characteristic p, it is shown that two abelian p-groups have isomorphic algebras only if they are isomorphic; using a result of Perlis and Walker [ibid. **68** (1950), 420–426; MR **11**, 638], abelian $\mathfrak{G}$ and $\mathfrak{G}'$ have isomorphic algebras if and only if their p-Sylow groups $\mathfrak{P}$ and $\mathfrak{P}'$ are isomorphic, and $\mathfrak{G}/\mathfrak{P}$ and $\mathfrak{G}'/\mathfrak{P}'$ have isomorphic algebras. The proof rests on relating the degrees of nilpotency of elements of the radical of $\mathfrak{A}$ to the orders of group elements.

R. C. Lyndon (Ann Arbor, Mich.).

Pinsker, A. G. Partially ordered groups of countable type. Leningrad. Gos. Ped. Inst. Uč. Zap. **89** (1953), 9–18. (Russian)

A commutative group X is called a K-group if it admits a partial ordering compatible with the group operation such that every element has a non-negative majorant and every bounded set has a least upper bound. If every family D of bounded, pairwise disjoint elements of X is countable, then X is said to be of countable type ($X\in T_0$). If every such family D has cardinal number $\leqq\aleph_\alpha$ and $X\notin T_\beta$ $(\beta<\alpha)$, then X is said to be of type α ($X\in T_\alpha$). Theorem: If $X\in T_\alpha$, then every bounded completely ordered snbset of X contains at most $\aleph_\alpha$ elements. Theorem: Let $X\in T_\alpha$. If $E\subset X$ and E is bounded, then there is a set $F\subset E$ such that $\sup F=\sup E$ and F contains at most $\aleph_\alpha$ elements. For $\alpha=0$, the conditions of both of these theorems are also sufficient for X to be of countable type. Another concept of the same general type is also introduced and studied. *E. Hewitt* (Seattle, Wash.).

Pinsker, A. G. Regular and completely regular partially ordered groups. Leningrad. Gos. Ped. Inst. Uč. Zap. **89** (1953), 19–35. (Russian)

[For terminology, see the preceding review.] Let X be a K-group. Suppose that for every sequence $\{E_n\}$ of subsets of X such that

$$\lim_{n\to\infty}\ [\sup E_n]=x,$$

there exist finite subsets E_n' of E_n such that

$$\lim_{n\to\infty}\ [\sup E_n']=x.$$

Then X is said to be regular. This axiom has been studied previously for K-spaces [Kantorovič, Vulih, and Pinsker, Functional analysis in partially ordered spaces, Gostehizdat, Moscow-Leningrad, 1950; MR **12**, 340] and the author first points out that many of the properties of regular K-spaces also hold for regular K-groups. Theorem 1: A regular K-group is of countable type. Theorem 2: Let Y be a subgroup of a regular K-group such that $y\in Y$ and $0\leqq x\leqq y$ imply $x\in Y$. Then Y is regular. Conditions are also studied under which the smallest K-space containing a given regular K-group is regular.

E. Hewitt (Seattle, Wash.).

Stolt, Bengt. Abschwächung einer klassischen Gruppendefinition. Math. Scand. **3** (1955), 303–305 (1956).

Let A be a semigroup with an element d such that, for all a, there exist a unique x satisfying $ax=d$, and such that, for all b, there exists at least (at most) one y satisfying $yb=d$. Then A is a group. *R. C. Lyndon.*

Berman, S. D. p-adic ring of characters. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 583–586. (Russian)

Let G be a finite group and K a field of characteristic zero. The trace-functions corresponding to the representations of G over K are called K-characters of G. It is proved that every K-character χ can be represented in the form

$$\chi=\sum_i a_i\psi_i^*,$$

where each ψ_i is a K-character of some K-elementary subgroup E_i of G, ψ_i^* designates the K-character of G

induced by φ_i, and where the a_i are rational integers. Here, a K-elementary group is defined as a group E which is the product $E=HF$, where H is a cyclic normal subgroup of E, F is a p-group for some prime number p not dividing the order of H, and where the following condition concerning K is satisfied: If H is identified with the corresponding group of roots of unity in the algebraic closure $\bar{K}$ of K, then the automorphisms of H induced by the elements of F can be obtained by applying automorphisms of $\bar{K}/K$. The above theorem is a generalization of a theorem of R. Brauer who only considered "absolute" characters (i.e. $\bar{K}=K$) [see e.g. Brauer and Tate, Ann. of Math. (2) **62** (1955), 1–7; MR **16**, 1087; or Roquette, J. Reine Angew. Math. **190** (1952), 148–168; MR **14**, 844]; in this case a K-elementary group E is the direct product $E=H\times F$ and hence is elementary in the sense of Brauer. The method employed by the author determines the structure of the p-adic ring of K-characters, generalizing the corresponding results of Roquette for "absolute" characters (loc. cit.). — It may be mentioned that the results of the author can also be found in chap. I of Witt, J. Reine Angew. Math. **190** (1952), 231–245 [MR **14**, 845]. *P. Roquette* (Hamburg).

Curzio, Mario. I gruppi finiti che sono somma di tre o quattro laterali di sottogruppi propri. Boll. Un. Mat. Ital. (3) **10** (1955), 228–232.

The author proves that a finite group is the sum of three cosets of proper subgroups (not all the same), in the sense that each element belongs to exactly one of the cosets, if and only if the group, without being cyclic of order 4, possesses a subgroup of index 2, and that it is the sum of four cosets of proper subgroups (not all the same) if and only if, without being cyclic of order 4 or 9, it contains a subgroup of index 2 or 3. *D. G. Higman.*

Furstenberg, Harry. The inverse operation in groups. Proc. Amer. Math. Soc. **6** (1955), 991–997.

The author gives an elegant set of postulates for groups in terms of a single binary operation which occurs quite frequently in group theoretic analyses, ab^{-1}.

Let G be a system with an operation ab such that (1) $a\circ b\in G$ for any $a, b\in G$, (2) $(a\circ c)\circ(b\circ c)=a\circ b$ for any $a, b, c\in G$, (3) $a\circ G=G$ for any $a\in G$. Then it follows that there is an $e\in G$ such that $a\circ a=e$ for all $a\in G$, that G is a group under the operation $ab=a\circ(e\circ b)$, and that $a\circ b=ab^{-1}$. If in addition $(c\circ b)\circ(c\circ a)=a\circ b$ for all $a, b, c\in G$, then G is abelian.

In analogy with semi-groups, a "half-group" is a system G satisfying (1) and (2). (Not every half-group is a group.) A structure theorem for half-groups is demonstrated.

W. W. Boone (Princeton, N.J.).

Lazard, Michel. Lois de groupes et analyseurs. Ann. Sci. Ecole Norm. Sup. (3) **72** (1955), 299–400.

Let U be a set; for any $n>0$, let $\prod_n U$ be the product of n sets identical to U, and let $A^n(U)$ be the set of mappings of $\prod_n U$ into U. Then the operation of composition of mappings introduces an algebraic structure in the set $A=\bigcup_{n>0} A^n(U)$: if $f\in A^m(U)$ and $g_i\in A^n(U)(1\leqq i\leqq m)$, there is given an element $(T_{mn}f)(g_1, \cdots, g_m)=f(g_1, \cdots, g_m)$ of $A^n(U)$ which maps the element $(x_1, \cdots, x_n)$ upon $f(g_1(x_1, \cdots, x_n), \cdots, g_m(x_1, \cdots, x_n))$. The author gives an abstract characterization of this operation of composition by introducing a new type of algebraic structure, the compositor: a compositor is defined by the data of an infinite sequence $(A_n)_{n>0}$ of sets, of a family of laws of composition T_{mn} (where T_{mn} maps $A^m\times(A^n\times\cdots\times A^n)$, with m factors A^n, into A^n) and of a family of distinguished elements $e_{mi}\in A^m$ $(1\leqq i\leqq m)$ (which, concretely, correspond to the projections of $\prod_m U$ upon its various factors), these data being linked together by axioms (which would be too long to specify here); these axioms are such that any abstractly given compositor may be interpreted as a subset of $\bigcup_{n>0} A^n(U)$ for a suitable choice of the set U.

The notion of an analyzer is a special case of the notion of compositor. While the set U introduced above was an arbitrary set, we consider now the case where U is a graded module (over some commutative ring Ω) and the class of mappings we want to study is such that any mapping of the class admits a decomposition into homogeneous components. In order to take these new factors into consideration, it is postulated that, for each $n>0$, the nth set A^n (in the terminology used above) is n-graded, i.e. is decomposed into a direct sum $\sum_\alpha A_\alpha{}^n$, where α runs over the systems $(\alpha_1, \cdots, \alpha_n)$ of n integers >0; from the point of view of concrete interpretation, $A_\alpha{}^n$ will stand for the set of mappings of $\prod_n U$ into U which are homogeneous of degree α_i with respect to their ith argument $(1\leqq i\leqq n)$; new axioms are introduced concerning this gradation of the A^n; here again, these axioms are such that any abstractly given analyzer may be interpreted as a set of mappings $\prod_n U\to U$ (for all $n>0$) where U is a module, the gradation in the analyzer corresponding to a decomposition of mappings of this set into their homogeneous components.

One may define for analyzers the usual algebraic notions: subanalyzers, homomorphisms, restrictions and extensions of the basic ring, $\cdots$. In particular, one has the notions of systems of generators of an analyzer, of free analyzer and of an analyzer which is defined by generators and relations. The analyzer A generated by a single element of A^2, subjected to the conditions of being bilinear (i.e. homogeneous of degree (1,1)), associative and commutative is called the classical analyzer; it governs these functions of n arguments in a commutative ring which may be expressed in terms of addition and multiplication only.

Let A be an analyzer which is represented as a class of mappings $\prod_n U\to U$, where U is a graded module. Any mapping F of $\prod_n (U\times U)$ into $U\times U$ may be regarded as a pair (f, g), where f and g are mappings of $\prod_{2n} U$ into U; the set of those F for which f and g belong to A is an analyzer, which is called the Cartesian square of A. More generally, one may define the Ith power of an analyzer for any set I of indices.

If A is an analyzer, $f\in A^m$, $g_1, \cdots, g_m\in A^n$, the expression of the homogeneous components of $f(g_1, \cdots, g_m)$ in terms of those of $f, g_1, \cdots, g_m$ is given by a generalized Taylor formula, which involves operators which depend on A and which generalize the partial differentiations for polynomials; these operators are called the Sanow operators of A.

The operation of composition has also a meaning for objects like formal power series (with constant term 0) which are not functions; they differ from functions essentially in this that it is allowed to form infinite sums of homogeneous elements provided the degrees of these elements are all distinct. To cover this case, the author introduces the notion of a complete analyzer: instead of assuming that A^n is a direct sum $\sum_\alpha A_\alpha{}^n$, one assumes that A^n is a product $\prod_\alpha A_\alpha{}^n$, and one adds an axiom of

continuity for the operators $(f, g_1, \cdots, g_m)\to f(g_1, \cdots, g_m)$. In any complete analyzer, one has a theorem of implicit functions which is a direct generalization of the one which holds for formal power series.

Let A be an analyzer; if $\alpha=(\alpha_1, \cdots, \alpha_n)$ is a system of n nonnegative integers, set $|\alpha|=\alpha_1+\cdots+\alpha_n$. For any $r>0$, let A_r^n be the set of elements of A^n whose components in the A_α^n are 0 whenever $|\alpha|\neq r$. The elements of A_r^n are said to be of total degree r. Every element of A^n may be decomposed into a sum (an infinite sum if A is complete) of homogeneous elements of degrees $0, 1, \cdots, r, \cdots$. If f and g are in A^n, we write $f\equiv g$ (mod total degree r) if the components of total degrees $<r$ of $f-g$ are all equal to 0.

A group law in a complete analyzer A is defined to be an element f of A^2 which has the following properties: $f(f(x, y), z)=f(x, f(y, z))$ and $f(x, y)\equiv x+y$ (mod total degree 2); we shall write xy instead of $f(x, y)$ (although the neutral element of the group law is the zero of the analyzer). Given any group law, there always exists an "inverse" function in the analyzer, i.e. an element x^{-1} of A^1 such that $xx^{-1}=x^{-1}x=0$; by iteration, one defines the "power" functions x^n, first for $n>0$ and then for any n by $x^{-n}=(x^{-1})^n$. The symbol $xyx^{-1}y^{-1}$ represents an element of A^2; the homogeneous component $[x, y]$ of degree 2 of this element is a Lie bracket, i.e. it is bilinear, and satisfies the Jacobi identity and the identity $[x, x]=0$.

If A is an analyzer, the elements of A^1 (functions of one argument) form a monoid under composition. Let f be a group law in A and φ an invertible element of A^1; if we set $f^\varphi(x, y)=\varphi(f(\varphi^{-1}(x), \varphi^{-1}(y)))$, then f^φ is a new group law, which is said to be equivalent to the law f. If φ is such that $\varphi(x)\equiv x$ (mod total degree 2), which implies that φ is invertible, then f^φ is said to be restrictedly equivalent to f. The main problem of the theory is to classify group laws relatively to equivalence (or restricted equivalence).

Let A be an analyzer, and A_r the module of homogeneous elements of degree r of A; A is called rational if, for every prime $p\leq r$, $f\to pf$ is an automorphism of the module A_r; this condition is always satisfied if the basic ring is a field of characteristic 0. The author proves that a group law in a rational analyzer is restrictedly equivalent to one and only one canonical group law, i.e. a law for which it is true that $x^n=nx$ for every n (this corresponds to the introduction of canonical coordinates in a Lie group). Assume now that the law xy is canonical; if $[x, y]$ is the Lie bracket associated to this law (cf. above), then $xy\equiv x+y+\frac{1}{2}[x, y]$ (mod total degree 3). It is proved that xy belongs to the rational subanalyzer generated by the element $[x, y]$. More generally, any group law is a homomorphic image of the universal Hausdorff group law, defined in the rational Lie analyzer which is generated (as a rational analyzer) by a single element $[x, y]$ submitted to the relations which express that it is a Lie bracket. This gives a precise formulation and a generalization of the Poincaré-Hausdorff theorem.

The situation is considerably more complicated in the case where the analyzer A is not rational. In that case, in order to construct group laws, one may try a method of successive approximations. If $f\in A^2$, let Γf be the element $f(f(x, y), z)-f(x, f(y, z))$ of A^3; is f called an r-bud if we have $\Gamma f\equiv 0$ (mod total degree $r+1$), $f\equiv x+y$ (mod total degree 2). Assume that this condition is satisfied; then we try to determine a homogeneous element h of degree $r+1$ of A^2 such that $f+h$ is an $(r+1)$-bud. This however is not always possible. The study of the conditions of possibility of this problem leads to the theory of the cohomology of an analyzer A. A coboundary operator δ is defined in A by

$$(\delta f)(x_1, \cdots, x_{n+1})=f(x_2, \cdots, x_{n+1})+\sum_{i=1}^{n}(-1)^i f(x_1, \cdots, x_i+x_{i+1}, \cdots, x_{n+1})+(-1)^{n+1}f(x_1, \cdots, x_n);$$

this leads to the definition of the cohomology groups of A. Returning to the problem formulated above, it turns out that, for any r-bud f, the homogeneous component of degree of $r+1$ of Γf is a 3-cocycle, and a necessary and sufficient condition for the existence of an h with the required property is that this cocyle be cohomologous to 0. Now, let f and g be group laws; assume that they determine the same r-bud, i.e. that we have $f\equiv g$ (mod total degree $r+1$). In order to study the equivalence problem, one tries to determine a homogeneous element a of degree $r+1$ of A^1 in such a way that $f^\varphi\equiv g$ (mod total degree $r+2$), where φ stands for $x+a(x)$ and f^φ is defined as above. It turns out that, in any case, the homogeneous component of degree $r+1$ of $f-g$ is a 2-cocycle, and that our problem is possible if and only if this 2-cocycle is cohomologous to 0. In order to obtain more definite results, the author makes restrictive assumptions both on the type of group laws to be considered and on the analyzer A: he assumes that we are dealing with commutative group laws and that A is the qth Cartesian power of the classical analyzer (or rather of its completion): this corresponds to the case of the formal Lie groups studied by Bochner and Dieudonné. In this type of analyzers, one is able to determine explicitly all symmetric 2-cocycles; making use of this fact, the author proves that, if the basic ring Ω has no torsion, every commutative r-bud may be extended all the way through, i.e. to a group law. Moreover, for any number q of parameters, he constructs a universal commutative group law; this is a commutative group law f in the qth power of the classical analyzer over a ring A (which is a polynomial ring in countably many unknowns over the integers) with the property that, for every commutative ring Ω and for any group law f' in the qth power of the classical analyzer over Ω, there is a uniquely determined homomorphism of A into Ω which transforms the group law f into the law f'. This shows that the commutative group laws over any ring may be described by free parameters varying over Ω; but it does not solve the equivalence problem for these laws.

The author also obtains a number of results on the cohomology of any analyzer A. The cohomology groups $H^n(A)$ are graded by means of the total degree in A; let $H_r^n(A)$ be the set of homogeneous elements of degree r in $H^n(A)$. Then $H_r^n(A)$ is $\{0\}$ if $r<n$; if $r>n$, it is a torsion group whose torsion involves only prime numbers $<r$; there is a homomorphism of the module of n-linear skew symmetric elements of A^n into $H_n^n(A)$ whose kernel and cokernel are torsion groups whose torsion involves only primes $\leq n$. Some results are given concerning the cohomology of a Cartesian power of an analyzer, leading to the complete determination of this cohomology in dimensions 1 and 2. A study is made of the cohomology of a classical analyzer. Finally, it is shown that any biadditive element of A^2 may be used to define a product in cohomology.

A number of "proofs" of results contained in this paper reduce to a few indications on how a highly specialized and omniscient reader might be able to construct a proof. This way of doing was probably forced upon the author once he had decided to include so much material in a single paper. Fortunately definitions and statements of results are always perfectly accurate. *C. Chevalley.*

Schenkman, Eugene. A certain class of semigroups. Amer. Math. Monthly **63** (1956), 242–243.

Let S be a cancellation semigroup in which the equation (1) $ab=xba$ is solvable for every couple $a, b \in S$. It is pointed out that such semigroups have many group properties. This is due to the fact that the following notions can be introduced: i) commutator of a with b (i.e. the solution $x=[a, b]$ of (1)), ii) the conjugate of b by a (i.e. the element $[a, b]b \in S$), iii) normal subsemigroup (i.e. a set N having the property $[s, n]n \in N$ for all $n \in N$, $s \in S$). Main theorem: S contains a subgroup G_* (the set of all the commutators) and S is contained in a supergroup G^* such that G^*/G_* is abelian. *Š. Schwarz.*

See also: Berman, p. 1048; Hattori, p. 1119; Hewitt and Zuckerman, p. 1048; Vosper, p. 1056.

THEORY OF NUMBERS

Kubilyus, I. P.; and Linnik, Yu. V. An elementary theorem on the theory of prime numbers. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 2(68), 191–192. (Russian)

On the extended Riemann Conjecture for the Hecke ζ-function of the Gaussian field it is true that there exists an infinite number of primes p representable as k^2+l^2, where k and l are integers such that $l=O(\ln p)$. In contrast the authors give a completely elementary and exceedingly simple proof of the weaker result that there exist infinitely many pairs of primes p_1, p_2 such that $p_1p_2=k^2+l^2$ with k and l integral and $l=O(\ln p_1p_2)$. The argument amounts to little more than the application of Dirichlet's 'Schubfachprinzip' to a suitable division of a circle in the complex plane into equal sectors. *H. Halberstam.*

Hercigonja, Mira. Le tableau des nombres premiers. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. **10** (1955), 183–188. (Serbo-Croatian summary)

If one writes the natural numbers in a triangular array in which 1 2 3 constitute the first row, 4-8 the second, 9-15 the third etc. the positions occupied by the primes form an interesting pattern. Primes belonging to a quadratic formula, like Euler's x^2+x+17, x^2+x+41, now lie on straight lines of slopes ± 1 and reveal themselves graphically by their high frequency of primes. Legendre's $2x^2+29$ does not show up so well, the corresponding points being on a parabola. Other formulas suggested to the author by the drawings given in the paper are x^2+x+A with $A=101$, 107, 221 and 227. The formulas contain many primes. *D. H. Lehmer.*

Browkin, Georges; et Schinzel, André. Sur les nombres de Mersenne qui sont triangulaires. C. R. Acad. Sci. Paris **242** (1956), 1780–1781.

It is proved that the only solutions in positive integers of $2^x-1=y(y+1)/2$ are (1, 1), (2, 2), (4, 5) and (12, 90). *I. Niven* (Eugene, Ore.).

Obláth, Richárd. Une tentative pour reconstruire une démonstration de Fermat. Mat. Lapok **4** (1953), 18–30. (Hungarian. Russian and French summaries)

The author discusses the methods by which Fermat might have proved some of his statements in number theory. In particular he suggests that Fermat might have been in possession of the so-called third proof of Euler of the congruence $a^p\equiv a \pmod{p}$, this proof utilised group theoretic considerations. *P. Erdös* (Jerusalem).

Larras, Jean. Sur la primarité des nombres de Fermat. C. R. Acad. Sci. Paris **242** (1956), 2203–2204.

The author proves that every factor of the Fermat number $F_n=2^{2^n}+1$ is congruent to 1 modulo 2^{n+2}, a theorem given by Lucas in 1878 [see L. E. Dickson, History of the theory of numbers, v. 1, Carnegie Inst. Washington, 1919, p. 376]. The author points out a fact about Fermat numbers F_n when $n=2^\mu-2=14$. For $p=k\cdot 2^{n+2}+1$ to divide F_n it is necessary and sufficient that p divides k^m+1 where $m=2^{n-\mu}$. Thus $\mu=4$ steps in the successive squarings and reduction modulo p are saved in testing F_{14}. However in dealing with powers of 2 instead of powers of k one may start with 2^{16} so that 4 steps are eliminated here too. Thus the reviewer fails to see why the calculations are "accelerated considerably." The author has tested all possible primes <131072001 as factors of F_{14} without success. This partially confirms a result of Selfridge [Math. Tables Aids Comput. **7** (1953), 274–275] that F_{14} has no factor $<2^{32}$. *D. H. Lehmer.*

Volkmann, Bodo. Ein Satz über die Menge der vollkommenen Zahlen. J. Reine Angew. Math. **195** (1955), 152–155 (1956).

Let $\sigma(n)$ denote the sum of the divisors of the positive integer n. Let $A(x)$ $[V(x)]$ denote the number of $n\leqq x$ with $n|\sigma(n)$ $[\sigma(n)=2n]$ and let $V_i(x)$ be the number of odd $n\leqq x$ with $\sigma(n)=2^in$ $(i=1, 2, \cdots)$. Kanold proved $A(x)=o(x)$ [same J. **194** (1955), 218–220; MR **17**, 238]. The author proves $V(x)=O(x^{5/6})$ and $V_i(x)=O(x^{1-\frac{1}{2}(i+2)^{-1}})$. His first result is already superseded by Hornfeck's $V(x)<x^{\frac{1}{2}}$ [Arch. Math. **6** (1955), 442–443; MR **17**, 460]. *P. Scherk* (Kingston, Ont.).

Scherk, Peter. An inequality for sets of integers. Pacific J. Math. **5** (1955), 585–587.

Let $A=\{a\}$, $B=\{b\}$ be sequences of nonnegative integers, and let $C=A+B$. Using a method of Besicovitch [J. London Math. Soc. **10** (1935), 246–248] the author proves the following theorem:

Let $x \in A$ $(x=0, 1, \cdots, h;\ h\geqq 0)$, $0 \in B$ or $1 \in B$, $n \notin C$. Further $C(n)<A(n-1)+B(n)$. Then there exists an integer m satisfying $m \notin C$, $0<m<n-h-1$ such that

$$C(m, n)\geqq A(n-m-1)+B(m, n),$$

where $A(m, n)=\sum_{m<a<n} 1$ and $A(n)=A(0, n)$.

The author deduces several consequences from this result. *P. Erdös.*

Egan, M. F. The harmonic logarithm. Math. Gaz. **40** (1956), 8–10.

Let $x>0$, put $x=[x]+\alpha$, $0\leqq\alpha<1$. The author defines the harmonic logarithm $L(x)$ of x as

$$L(x)=L([x])=\sum_{r=1}^{[x]}\frac{1}{r}\quad (L(0)=0).$$

He proves various inequalities for $L(x)$ using only simple inequalities on rational numbers [see also Broderick, J. London Math. Soc. **14** (1939), 303–310; Proc. Roy. Irish. Acad. Sect. A. **46** (1940), 17–24; MR **1**, 41, 292]. *P. Erdös* (Jerusalem).

Schinzel, A.; et Sierpiński, W. Sur l'équation $x^2+x+1=3y^2$. Colloq. Math. **4** (1956), 71–73.

All solutions of $x^2+x+1=3y^2$ in positive integers are

given by the recursion formulas $x_{n+1}=7x_n+12y_n+3$, $y_{n+1}=4x_n+7y_n+2$, with $x_1=y_1=1$. *I. Niven.*

Fjellstedt, Lars. Einige Sätze über lineare Kongruenzen. Ark. Mat. 3 (1956), 271–274.

Let D be the area of the hyperplane $\sum_{i=1}^r a_ix_i=0$ which lies inside the cube $|x_i|\leq m^{1/r}$, where m and the a_i are integers. Let $A_r(m)$ be the number of solutions of $\sum_{i=1}^r a_ix_i\equiv 0 \pmod m$ lying inside the cube. Then

$$A_r(m)=D/(\sum_{i=1}^n a_i^2)^{\frac{1}{2}}+O(m^{(r-2)/r}).$$

A similar result is proved for systems of congruences. These results relate to earlier work [cf. Brauer and Reynolds, Canad. J. Math. 3 (1951), 367–374; MR 14, 21]. *I. Niven* (Eugene, Ore.).

Carlitz, Leonard. An extension of Bauer's congruence. Math. Nachr. 14 (1955), 183–191.

In the first part of this paper the author extends in various directions the well-known congruence of Bauer [Nouvelles Ann. Math. (4) 2 (1902), 256–264]. One of his main results may be stated as follows. Let p be an odd prime and form the product

$$Q(z)=\prod(z+r_1u_1+\cdots+r_ku_k),$$

where all the r_i range from 0 to p^m-1 and

$$(r_1,\cdots,r_k,p)=1.$$

Then

$$Q(z)\equiv\left\{\frac{\Delta(u_1,\cdots,u_k,z)}{z\Delta(u_1,\cdots,u_k)}\right\}^{p^{k(m-1)}} \pmod{p^m},$$

where $\Delta(x_1,\cdots,x_k)$ is the determinant $|x_i^{p^j}|$ $(1\leq i\leq k,$ $0\leq j\leq k-1)$. The remainder of the paper is devoted to applications. An example is: If

$$Q(z)=z^n+C_1z^{n-1}+\cdots+C_n,$$

where $n=p^{km}-p^{k(m-1)}$, then $C_r\equiv 0 \pmod{p^m}$ for $p-1\nmid r$; moreover $C_{2r+1}\equiv 0 \pmod{p^{2m-1}}$ for all r and $C_{2r+1}\equiv 0 \pmod{p^{2m}}$ for $p-1\nmid 2r$. *A. L. Whiteman.*

Vosper, A. G. The critical pairs of subsets of a group of prime order. J. London Math. Soc. 31 (1956), 200–205.

Let $A=\{\alpha\}$, $B=\{\beta\}$ denote non-empty subsets of the additive group G of prime order p. The set $A+B$ consists of all the elements $\alpha+\beta$. Let $n(A),\cdots$ denote the number of elements of $A,\cdots$. A theorem by Cauchy and Davenport states that $n(A+B)\geq\min(p, n(A)+n(B)-1)$ [Davenport, same J. 10 (1935), 30–32]. The author proves that equality holds if and only if A and B satisfy one of the following conditions:

(i) $n(A)+n(B)>p$; (ii) $\min(n(A), n(B))=1$;

(iii) there is a $\gamma\subset G$ such that $A=\{\gamma-\bar\beta\}$ where $\bar\beta$ ranges through the complement of B in G; (iv) A and B are arithmetical progressions with the same common difference. *P. Scherk.*

de Bruijn, N. G. On number systems. Nieuw Arch. Wisk. (3) 4 (1956), 15–17.

A collection of sets $S_1, S_2, \cdots$ of nonnegative integers, each including zero and at least one other integer, is called a number system if each nonnegative integer x can be uniquely decomposed as $x=s_1+s_2+\cdots$, where $s_i\in S_i$. A special case, called by the author a British number system, is determined by an infinite sequence of integers $g_i>1$, where S_i consists of the numbers $ng_1g_2\cdots g_{i-1}$ for $0\leq n<g_i$; a further special case is the Continental number system where each $g_i=10$. The sets S_i of a number system may be grouped together in batches of different sizes to form a new number system called a degeneration of the original number system. The author proves that every number system is a degenerated British number system. This follows from repeated applications of a lemma which states that if $S_1, S_2, \cdots$ is a number system of at least two sets, then it is a degeneration of a number system $T_1, T_2, \cdots$ to which there belongs an integer $g>1$ such that T_1 consists of the elements $0, 1, 2, \cdots, g-1$ only, and T_2 contains only multiples of g. *R. A. Rankin* (Glasgow).

Linnik, Yu. V.; and Malyšev, A. V. Applications of the arithmetic of quaternions to the theory of ternary quadratic forms and to the decomposition of numbers into cubes. Amer. Math. Soc. Transl. (2) 3 (1956), 91–162.

Translated from Uspehi Mat. Nauk (N.S.) 8 (1953), no. 5(57), 3–71; 10 (1955), no. 1(63), 243–244; MR 16, 450.

Nagell, Trygve. Sur quelques problèmes dans la théorie des restes quadratiques et cubiques. Ark. Mat. 3 (1956), 211–222.

Let $\pi(p,n)$ and $\psi(p,n)$ denote the smallest positive prime n-ic residue and non-residue respectively modulo p; let $\pi^*(p,n)$ and $\psi^*(p,n)$ denote the smallest odd primes with the same properties. Say that the density of a set A of primes is $\liminf A(x)/\pi(x)$, where $A(x)$ denotes the number of primes in A which are $\leq x$ and $\pi(x)$ is the number of primes $\leq x$. Write p_n for the nth prime. For $n>1$ the density of primes p such that $\psi^*(p,2)=p_n$ is $1/2^{n-1}$; similarly for $\pi^*(p,2)=p_n$. For fixed n the density of primes p such that $\psi(p,3)=p_n$ is positive; similarly for $\pi(p,3)=p_n$. Necessary and sufficient conditions that 2 (or 3 or 5 or 7) be a cubic residue of a prime p of the form $(x^2+27y^2)/4$ are given in terms of simple divisibility conditions on x and y. The density of primes enjoying the property is shown to be 1/6 (or 1/6 or 1/4 or 1/4). From this it follows that the density of primes having 2 (or 3 or 5 or 7) as a cubic non-residue is 1/3 (or 1/3 or 1/4 or 1/4). Several of the proofs employ results in the field $R(\varrho)$, where ϱ is a primitive cube root of unity. The author has done earlier work on similar problems [Ark. Mat. 1, 185–193 (1950), 573–578, 579–586 (1952); MR 11, 640; 14, 247, 248]. *I. Niven* (Eugene, Ore.).

Fjellstedt, Lars. A theorem concerning the least quadratic residue and non-residue. Ark. Mat. 3 (1956), 287–291.

Let $\psi^*(p,2)$ and $\pi^*(p,2)$ denote the smallest odd prime quadratic non-residue and quadratic residue respectively modulo p. The author claims to prove that $\psi^*(p,2)<6\log p$ and $\pi^*(p,2)<6\log p$ for p sufficiently large. The proof is incorrect (specifically, in the arguments after (4) on page 289). *P. T. Bateman* (Princeton, N.J.).

van der Blij, F. Quadratic forms and Euler products. Nederl. Akad. Wetensch. Proc. Ser. A. 59=Indag. Math. 18 (1956), 229–237.

The zeta-functions associated with the representations of a number by a quadratic form in an even number of variables do not, in general, have Euler products. This remains true if one takes means over all classes of a fixed genus or over all genera with a fixed discriminant. How-

ever, by taking linear combinations of Dirichlet series with different discriminants one can obtain Euler products, and this the author does in various cases. *R. A. Rankin.*

See also: Froda, p. 1121; Porter, p. 1140; Sanchez-Mazas, p. 1037.

Analytic Number Theory

Val'fiš, A. Z. On the representation of numbers by sum of squares. Asymptotic formulas. Amer. Math. Soc. Transl. (2) **3** (1956), 163–248.

Translated from Uspehi Mat. Nauk (N.S.) **7** (1952), no. 6(52), 97–178; MR **15**, 936.

Roux, Delfina. Sui numeri primi delle progressioni aritmetiche. Boll. Un. Mat. Ital. (3) **11** (1956), 55–63.

Appealing only to the classical Tshebysheff results in prime number theory the author derives a number of theorems about primes in an arithmetic progression. In particular, he is able to establish in this manner the following two results. (1) If a is one of the integers 4, 5, 6, 8, 9, 10, 12, 15, 18, 30 and if b is relatively prime to a, then the number of primes of the form $ax+b$ which do not exceed ξ has the order of magnitude $\xi/\log \xi$. (2) Let (a, η) be one of the pairs (6, 3/4), (8, 1/5), (12, 1/2), (30, 1/4) Then for each integer ξ greater than a suitable ξ_0, the set of integers $ax+b$, $(a, b)=1$, $x=[\eta\xi]+1, \cdots, \xi$ contains at least one prime. *A. L. Whiteman.*

Knödel, Walter. Primzahldifferenzen. J. Reine Angew. Math. **195** (1955), 202–209 (1956).

Put $d_n=p_{n+1}-p_n$. The reviewer and Rényi proved [Simon Stevin **27** (1950), 115–125; MR **11**, 644] that

$$c_1\frac{x}{(\log x)^2}<\sum_{p_n<x}\frac{1}{d_n}<c_2\frac{x}{(\log x)^2}\log\log x.$$

In 1951 the author and Schmetterer observed (unpublished) that

$$c_3\frac{x}{(\log x)^2}<\sum_{p_n<x}\left|\frac{1}{d_n}-\frac{1}{d_{n+1}}\right|<c_4\frac{x}{(\log x)^2}.$$

Using Bruns method the author proves the following theorems:

$$c_5\sum_{p_n<x}\frac{1}{d_n}<\sum_{p_n<x}\left|\frac{1}{d_n}-\frac{1}{d_{n+1}}\right|<c_6\sum_{p_n<x}\frac{1}{d_n}$$

and

$$c_7\frac{x}{(\log x)^2}<\sum{}'\frac{1}{d_n}<c_8\frac{x}{(\log x)^2},$$

where the — indicates that $p_n<x$ and $d_n\geqq d_{n-1}$.

P. Erdös (Haifa).

Šidlovskiĭ, A. B. On a new criterion of the transcendentality and algebraic independence of values assumed by a class of entire functions. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 399–400. (Russian)

Continuing the authors' earlier work on E-functions [for references and definitions see same Dokl. (N.S.) **105** (1955), 35–37; MR **17**, 947], the following theorems are announced. I. Let the set of E-functions $f_1(z), \cdots, f_m(z)$ be solutions of a system of m linear differential equations of first order

$$y_k'=Q_{k0}(z)+\sum_{i=1}^{m}Q_{ki}(z)y_i\ (k=1, \cdots, m)$$

whose coefficients $Q_{ki}(z)$ are rational functions of z with algebraic numerical coefficients, and let α be an arbitrary algebraic number different from the zeros and poles of all $Q_{ki}(z)$. Then in order that the l numbers $f_1(\alpha), \cdots, f_l(\alpha)$, $1\leqq l\leqq m$, be algebraically independent, it is necessary and sufficient that the functions $f_1(z), \cdots, f_l(z)$ be algebraically independent over the field $R(z)$ of rational functions of z. In particular, the numbers $f_i(\alpha)$, $i=1, \cdots, m$, are transcendental if and only if the corresponding functions $f_i(z)$ are transcendental. II. Under the same hypotheses, if the functions $f_1(z), \cdots, f_l(z)$, $2\leqq l\leqq m$ are algebraically independent over $R(z)$, then the $l-1$ numbers $f_i(\alpha)/f_s(\alpha)$, $i=1, \cdots, l$, $i\neq s$, are algebraically independent, for $s=1, \cdots, l$.

These theorems contain all the author's previous results. *W. J. LeVeque* (Ann Arbor, Mich.).

Carlitz, Leonard. Arithmetic properties of elliptic functions. Math. Z. **64** (1956), 425–434.

Put

$$\operatorname{sn} x=\operatorname{sn}(x, u)=\sum_{m=0}^{\infty}A_{2m+1}(u)x^{2m+1}/(2m+1)!,$$

where $u=k^2$ in the usual notation; then $A_{2m+1}(u)$ is a polynomial in u with integral coefficients. Also put $x/\operatorname{sn} x=\sum_{m=0}^{\infty}\beta_{2m}(u)x^{2m}/(2m)!$, where $\beta_{2m}(u)$ is a polynomial with rational coefficients. By making use of the Fourier expansions for the Jacobi elliptic functions the author derives results of the following two types. (1) If p is an odd prime, $p^{e-1}(p-1)|t$ and r is an arbitrary integer $\geqq 1$, then

$$\sum_{s=0}^{r}(-1)^{r-s}\binom{r}{s}A_{p^{(r-s)t/(p-1)}}A_{n+st}\equiv 0\ (\operatorname{mod}(p^n, p^{er})).$$

(2) The polynomials $\beta_{2m}(u)$ satisfy the congruence $p\beta_{2m}(u)\equiv -A_{p^{2m/(p-1)}}(u)$ or 0 (mod p) according as $p-1|m$ or not. The first result is a so-called Kummer congruence, whereas the second is an analogue of the Staudt-Clausen theorem. The proofs are based upon the corresponding properties of the Bernoulli and Euler numbers. *A. L. Whiteman* (Los Angeles, Calif.).

Petersson, Hans. Über die arithmetischen Eigenschaften eines Systems multiplikativer Modulfunktionen von Primzahlstufe. Acta Math. **95** (1956), 57–110.

The Fourier coefficients of many modular forms may be interpreted as the number of partitions of the corresponding index, with the number, or size (or both) of the summands subjected to specific restrictions. Hence, any progress made concerning the Fourier expansions of such modular forms, can be used, in principle, in order to obtain corresponding information relative to certain partition functions. Using results and methods of his previous papers [Math. Ann. **127** (1954), 33–81; Abh. Deutsch Akad. Wiss. Berlin. Math. Allg. Nat. **1954,** no. 2; see also S. B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1950,** 417–494; MR **15,** 686; **17,** 129; **12,** 806], the author obtains representations by finite linear combinations of absolutely convergent series for the Fourier coefficients of certain modular functions and forms. Identifying the dominant terms of these expansions, he obtains asymptotic formulae for these Fourier coefficients, i.e. for certain partition functions. Specifically, the following three

partition problems are considered: Problem I. Let q be a prime and denote by k_0, k_1 two non-negative integers, $k=k_0+k_1>0$, by $\eta(\tau)$ the Dedekind η-function, set $u_m=\exp(2\pi i\tau/m)$ and $\beta_\infty=((q-1)/24)(-k_0+k_1(q-1))$. Then

$$F(\tau)=\eta^{k_0+2k_1}(\tau)\eta^{-k_0-k_1}\left(\frac{\tau}{q}\right)\eta^{-k_1}(q\tau)=\sum_{n=0}^{\infty}r_n(k_0,k_1,q)u_q^{\,n-\beta_\infty}.$$

Here $r_n=r_n(k_0, k_1, q)$ represents the number of partitions of the integer n into summands not divisible by q, subject to the following restrictions: the summands occur in the k colors; each summand may appear arbitrarily often in any of the first k_0 colors, but only $q-1$ times in any of the last k_1 colors and two partitions are considered identical if each summand occurs in both of them the same number of times in each color. The study of the function $F(\tau)$ then leads to the desired expansions and asymptotic formulae for $r_n(k_0, k_1, q)$. Problem II. Let the prime $q>5$ satisfy $q\equiv 1 \pmod 4$, let k^+ and k^- be non-negative integers, $k^++k^->0$. Let the integer m occur in any of the k^+ colors if $(m/q)=+1$ and in any of the k^- colors, if $(m/q)=-1$. Then $\pi_n(\mathfrak{k}, q)$ stands for the number of partitions of n into summands m not divisible by q, the colors in which they may occur being restricted only by the value of the Legendre symbol (m/q), and the identity of two partitions being defined as in Problem I. These partitions appear as Fourier coefficients of modular functions, invariant under the transformations of a subgroup $\Gamma^{0+}[q]$ of the modular group, defined as follows:

$$L=\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\Gamma^{0+}[q]\subset\Gamma \text{ if } \left(\frac{\alpha}{q}\right)=+1 \text{ and } \beta\equiv 0 \pmod q.$$

Problem III is similar to Problem II, but in addition to summands $m\not\equiv 0 \pmod q$, appearing in k^++k^- colors, as in Problem II, also summands $m\equiv 0 \pmod q$ are admitted in k^0 distinct colors. The corresponding numbers $\varrho_n(\mathfrak{l}, q)$ of partitions of n appear as Fourier coefficients of modular forms of positive dimension $\frac{1}{2}k^0$, belonging to above defined subgroup $\Gamma^{0+}[q]$ of the modular group. The expressions obtained for $r_n(k_0, k_1, q)$, $\pi_n(\mathfrak{k}, q)$ and $\varrho_n(\mathfrak{l}, q)$ are too complicated to be reproduced here. Only two remarkable corollaries of Problem II [see, however, also the last paper mentioned above] will be quoted. Let $\pi_n^+(q)$ (and $\pi_n^-(q)$) stand for the number of partitions of n into summands m that are quadratic residues (or non-residues, resp.), mod q. Denote by $\pi_n^+(q, l)$ and $\pi_n^-(q, l)$ respectively, the corresponding number of partitions, when any summand may appear at most $l-1$ times ($l\geqq 2$). Let $a_5(q)$ stand for the number of representations of the prime $q>5$, $q\equiv 1 \pmod 4$, as a sum of five squares, set $a_5^*(q)=a_5(q)/240$ if $q\equiv 1 \pmod 8$, $a_5^*(q)=a_5(q)/560$ if $q\equiv 5 \pmod 8$. If the quadratic field over the rationals $P(q^{\frac{1}{2}})$ has fundamental unit $\varepsilon>1$ and class-number h, then, as $n\to\infty$,

$$\frac{\pi_n^+(q)}{\pi_n^-(q)}=\varepsilon^h\left\{1-\frac{\pi}{6}\sqrt{\left(3\left(1-\frac{1}{q}\right)\right)}a_5^*(q)n^{-\frac{1}{2}}+O(n^{-1})\right\},$$

and

$$\frac{\pi_n^+(l, q)}{\pi_n^-(l, q)}= 1+\frac{\pi}{6}\sqrt{\left(3\left(1-\frac{1}{q}\right)\left(1-\frac{1}{l}\right)\right)}(l-1)a_5^*(q)n^{-\frac{1}{2}}+O(n^{-1}).$$

Hence, if the number of equal summands is restricted ($<l$), then $\lim_{n\to\infty}\pi_n^+(l, q)/\pi_n^-(l, q)=1$; while, if no such restriction exists, $\lim_{n\to\infty}\pi_n^+(q)/\pi_n^-(q)=\varepsilon^h>1$. These results, proven here for $q>5$, generalize those obtained by Lehner [Duke Math. J. 8 (1941), 631–655; MR 3, 166] for $q=5$. *E. Grosswald* (Philadelphia, Pa.).

Koecher, Max. Zur Operatorentheorie der Modulformen n-ten Grades. Math. Ann. **130** (1956), 351–385.

Let Γ denote the modular group and let $\{\Gamma, k\}$ stand for the set of integral modular forms of negative dimension k. For every integer g, Hecke defined the operator $T(g)$ of $\{\Gamma, k\}$ into itself, as follows: If $f\in\{\Gamma, k\}$, then

$$f|T(g)=g^{-k-1}\sum_{\mathfrak{A}}(c\tau+d)^kf\left(\frac{a\tau+b}{c\tau+d}\right),$$

the sum being extended over all non-equivalent (under Γ) matrices $\mathfrak{A}=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ with integral elements and $ad-bc=g$.

The set of the $T(g)$ forms a commutative ring, whose structure is defined by

$$(1)\qquad T(g)T(h)=\sum_{d|(g,h)}d^{-k-1}T(gh/d^2).$$

As the Fourier coefficients of many modular forms are important number-theoretical functions, the Hecke operator is a very valuable tool in number theory. Also the Fourier coefficients of some modular forms in several variables have number-theoretical significance; therefore, the generalization of the Hecke operator to the case of modular of degree n is an important problem. Maass [Math. Ann. **124** (1951), 87–122; MR **13**, 823] defined the most general operator of this kind, in which the domain and the range of the operators are still the same. Formally, several further generalizations seem possible. The author is guided by heuristic considerations of the complex multiplication of abelian functions of n variables, as natural generalization of the complex multiplication of elliptic functions. Thus he is led to a set of rather general operators, forming again a ring. However, the generality achieved is paid for by a certain loss of simplicity. In the first place, the domain and range of the operators are distinct sets of functions; hence, the "inverse operator" is actually an operator of a different kind. Finally, two further operators are needed, that reduce to the identity operator, if $n=1$. Hence, four distinct operators are defined. The precise definition of these operators requires rather lengthy explanations and cannot be given in a few words. The operators $T(\mathfrak{H}, \mathfrak{H}\mathfrak{R})$ and $T^*(\mathfrak{H}\mathfrak{R}, \mathfrak{H})$ (the first argument stands for the domain of definition, the second for the range of the operator) however, satisfy relations generalizing (1) for the particular case $(g, h)=1$; the generalization of (1) for the general case $(g, h)>1$ is not obtained. The operators $V(\mathfrak{H}, \mathfrak{H}\mathfrak{G}^2)$ and $V^*(\mathfrak{H}\mathfrak{G}^2, \mathfrak{H})$ (same meaning of the arguments as for the T-operators) satisfy $V\cdot V^*=1$, while $V^*\cdot V\neq 1$ in general; but the operator $W(\mathfrak{H};\mathfrak{G})=\delta(H\cdot G^{-2})V^*(\mathfrak{H}, \mathfrak{H}\mathfrak{G}^{-2})V(\mathfrak{H}\mathfrak{G}^{-2}, \mathfrak{H})$ (here $\delta(\mathfrak{A})=1$ if $\mathfrak{A}$ is in canonical form, with integral, positive elementary divisors on the main diagonal; $\delta(\mathfrak{A})=0$ otherwise) acts upon the functions f_j of certain orthogonal bases of cusp forms like $f_j|W(\mathfrak{H};\mathfrak{G})=\chi_j(\mathfrak{H};\mathfrak{G})f_j$ with $\chi_j=0$ or 1. Finally, the "Euler product" decomposition of the Hecke operator,

$$T_s=\prod_p(1-p^{-s}T(p)+p^{-k-1-2s})^{-1},\quad T_s=\sum_{g=1}^{\infty}g^{-s}T(g)$$

can be generalized and an analogous relation holds for certain operators $T(\mathfrak{G})$, derived from the $T(\mathfrak{H}\mathfrak{G}^{-1}, \mathfrak{H})$.

E. Grosswald (Philadelpia, Pa.).

Herrmann, Oskar. Über den Rang der Schar der Spitzenformen zu Hilbertschen Modulgruppen. Math. Z. **64** (1956), 457–466.

The author considers the non-identical vanishing of the Poincaré series in complex variables τ_1, τ_2 for the Hilbert modular group for a real quadratic field of discriminant d. The transformation matrices are conjugates of a unimodular integral matrix M in $R(d^{\frac{1}{2}})$ with denominator $\gamma\tau+\delta$. With k a rational integer $\geqq 3$ and μ an integer of $R(d^{\frac{1}{2}})$, the series

$$G=\sum_{(\gamma,\delta)}\sum_{\lambda} N(\gamma\tau+\delta)^{-k}\exp 2\pi iS(\mu\lambda^2M\tau d^{-\frac{1}{2}})$$

is defined in the usual manner (to avoid repetitions in summation), with N and S as norm and trace respectively. In the Fourier expansion of G, the coefficients of $\exp 2\pi iS(\nu\tau d^{-\frac{1}{2}})$ for ν in $R(d^{\frac{1}{2}})$ break up into the combination of Kloosterman sums and Bessel functions. Estimates verify the non-identical vanishing of G when the principal term $(\gamma, \delta)=(0, 1)$ predominates as it does except for a finite set of k $(\leqq 9)$ and d $(\leqq 109)$. Similar results are shown to hold for ideal class generalizations of the group [see Gundlach, Acta. Math. **92** (1954), 309–345; MR **16**, 1000], and lower estimates can be made on the dimensionality of the cusp forms. *Harvey Cohn.*

Tomaševič, V. F. On a family of arithmetic normed systems of functions. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **13** (1954), 159–161. (Russian)

One of the examples given by N. P. Romanov [Izv. Akad. Nauk SSSR. Ser. Mat. **15** (1951), 131–152, p. 146, formula (30); MR **13**, 208] of a complete orthonormal system of functions over $L^2(0, \infty)$ concerned a one-parameter family $\varphi_n(x)$, here called $\Psi_n(s, x)$, where the parameter s satisfies $\operatorname{Re} s>\frac{1}{2}$. This is generalized by the author to a two-parameter family $\Psi_n(\alpha, s; x)$ periodic in the new parameter α with period 2π. This new family is also a complete orthonormal system and reduces to $\Psi_n(s, x)$ for $\alpha=0$. Detailed results are too complicated to quote here. There are several misprints. *R. A. Rankin.*

See also: Hasse, p. 947; Izumi, p. 1080; Lamprecht, p. 947; Lenz, p. 1082; Utiyama, p. 1163; van der Blij, p. 1056; van der Pol, p. 1082.

Algebraic Number Theory

Carlitz, L. The number of solutions of some equations in a finite field. J. Math. Soc. Japan **7** (1955), 209–223.

Earlier investigations by the author [Pacific J. Math. **3** (1953), 13–24; **4** (1954), 207–217; MR **14**, 848; **15**, 777] on the subject of the title are continued in the present paper. Making use of the well-known formulas for the number of solutions of $Q(\xi_1, \cdots, \xi_r)=\alpha$, where Q denotes a quadratic form with coefficients in $\mathrm{GF}(q)$, q odd, the author now considers equations of the type

$$(*)\qquad Q(\xi_1, \cdots, \xi_{2r+1})=g(\eta_1, \cdots \eta_s),$$

where $g(\eta)$ is a polynomial satisfying certain conditions. As a typical result the following may be cited. Let $g(\eta)=\prod_{i=1}^{s}(\eta_i^2+\beta_i\eta_i+\gamma_i)$ and suppose that $\beta_i^2-2\gamma_i\neq 0$ for $i=1, \cdots, s$. Then the number of solutions of (*) is given by $q^{2r+s}+(-1)^sq^r\psi((-1)^r\delta)$, where δ is the discriminant of $Q(\xi)$, and where $\psi(\alpha)=0, 1, -1$ according as α is 0, a square or a nonsquare of $\mathrm{GF}(q)$. *A. L. Whiteman.*

Hodges, J. H. The matric equation $AX=B$ in a finite field. Amer. Math. Monthly **63** (1956), 243–244.

The author derives a necessary and sufficient condition for the equation in the title to have solutions. He also determines the number of solutions when the equation is solvable. *A. L. Whiteman* (Los Angeles, Calif.).

Rogers, K. Indefinite binary hermitian forms. Proc. London Math. Soc. (3) **6** (1956), 205–223.

The author considers indefinite binary hermitian forms: $f(x, y)=ax\bar{x}+b\bar{x}y+\bar{b}x\bar{y}+cy\bar{y}$ with non-zero determinant $-d=ac-b\bar{b}<0$. He defines $M(f; x_0, y_0)$ to be the greatest lower bound of $|f(x+x_0, y+y_0)|$ taken over all Gaussian integers x, y; $M(f)$ is the sup $M(f; x_0, y_0)$ taken over all complex number pairs x_0, y_0. The author's main result is the following, under the definitions and notations above: $M(f)<2d^{\frac{1}{2}}/5$ except when $\pm f$ is equivalent to one of the following forms:

(i) $$f_1=d^{\frac{1}{2}}(e^{i\pi/4}\bar{x}y+e^{-i\pi/4}x\bar{y}),$$
(ii) $$f_2=(d/3)^{\frac{1}{2}}(x\bar{x}-3y\bar{y}),$$
(iii) $$f_3^{(1)}=d^{\frac{1}{2}}(x\bar{x}-y\bar{y}),$$
$$f_3^{(2)}=d^{\frac{1}{2}}(x\bar{y}+\bar{x}y),$$
(iv) $$f_4=(d/21)^{\frac{1}{2}}(x\bar{x}-21y\bar{y}),$$
(v) $$f_5^{(1)}=(2d/3)^{\frac{1}{2}}(x\bar{x}+(1+i)\bar{x}y/2+(1-i)x\bar{y}/2-y\bar{y}),$$
$$f_5^{(2)}=(d/6)^{\frac{1}{2}}(x\bar{x}-6y\bar{y}).$$

In each case the value of $M(f)$ is given and all points x_0, y_0 for which it is attained are given. If $|f(x, y)|$ assumes arbitrarily small non-zero values for Gaussian integers x, y, then $M(f)=0$.

Some of the details are omitted and reference made to the author's Ph. D. thesis [Cambridge, 1954]. *B. W. Jones* (Boulder, Colo.).

See also: Berman, p. 1048; Davenport and Roth, p. 1060; Kinohara, p. 1047.

Geometry of Numbers, Diophantine Approximation

Santaló, L. A. On geometry of numbers. J. Math. Soc. Japan **7** (1955), 208–213.

The author essentially proves the following lemma. Let m be a bi-invariant countably additive measure defined on a group G_1. Let F be a countable group of one-one transformations of G_1 onto itself leaving the measure invariant. Suppose that the unit x_0 of F is the identity transformation and let $x_1, x_2, \cdots$ be the other elements of F. Suppose that D_0 is a fundamental domain for G_1 with respect to F, which is measurable and which has a finite positive measure. Let $P_1, \cdots, P_N$ be N fixed elements in D_0. Define the element P_{hN+i} to be x_hP_i for $i=1, \cdots, N$ and $h=1, 2, \cdots$. Let f be a function defined so that $f(P_{j+N})=f(P_j)$, $j=1, 2, \cdots$. Then if H is a measurable set of G_1 the mean value over all x in D_0 of the sum $\sum f(P_i)$, taken over all P_i in xH is

$$\frac{m(H)}{m(D_0)}\sum_{i=1}^{N} f(P_i).$$

This lemma is used to prove two rather complicated theorems which reduce in one special case to results of Blichfeldt and Minkowski [see Blichfeldt, Trans. Amer. Math. Soc. **15** (1914), 227–235] and in another special case to results of M. Tsuji [J. Math. Soc. Japan **4** (1952),

189–193; MR 14, 623]. Other comparable generalizations of the theorems of Blichfeldt and Minkowski have been given by A. M. Macbeath [Thesis, Princeton, 1950] and by C. Chabauty [Bull. Soc. Math. France 78 (1950), 143–151, footnote 6; MR 12, 479]. *C. A. Rogers* (Birmingham).

Armitage, J. V. On a method of Mordell in the geometry of numbers. Mathematika 2 (1955), 132–140.

Let K be an n-dimensional star body with distance function $F(x_1, x_2, \cdots, x_n)$ and r an integer $1 \leq r \leq n$ for which (1) the adjoint of an automorph (a linear transformation $x_1 \to x_1', \cdots, x_n \to x_n'$ for which $F(x_1, x_2, \cdots, x_n) = F(x_1', x_2', \cdots, x_n')$ is an automorph. (2) Each point $(x_1, x_2, \cdots, x_n)$ can be mapped, by a suitably chosen automorph, into a point $(\alpha, \alpha, \cdots, \alpha, 0, 0, \cdots, 0)$, where r is the number of non-zero coordinates. K_{n-1} is the intersection of K by a certain hyperplane.

A lower bound is determined for the critical determinant $\Delta(K)$ in terms of $\Delta(K_{n-1})$ and a constant dependent on K. This result is applied to obtain two results due to Mordell, one involving the region $|x_1 \cdot x_2 \cdot \cdots \cdot x_n| \leq 1$ [Mat. Sb. N.S. 12(54) (1943), 273–276; MR 5, 201] and the second involving the n-dimensional sphere [J. London Math. Soc. 19 (1944), 6–12; MR 6, 57, 334].

The methods are used to obtain a result concerning more specialized star bodies. By means of this result an upper bound is found for the constant C_3 defined as follows. If C is a constant such that for any three real numbers $\theta_1, \theta_2, \theta_3$ not all rational, infinitely many rational approximations p_1/q, p_2/q, p_3/q exist with

$$\left(\theta_1 - \frac{p_1}{q}\right)^2 + \left(\theta_2 - \frac{p_2}{q}\right)^2 + \left(\theta_3 - \frac{p_3}{q}\right)^2 < \left(\frac{C}{q}\right)^{2/3},$$

then C_3 is the greatest lower bound of all the constants C.

D. Derry (Vancouver, B.C.).

Kasch, Friedrich. Abschätzung der Dichte von Summenmengen. II. Math. Z. 64 (1956), 243–257.

[For part I see Math. Z. 62 (1955), 368–387; MR 17, 712.] Let $A, B, \cdots$ denote sets of nonnegative integers. The set $\bar{A}$ consists of those nonnegative integers which do not lie in A. The set $C = A + B$ consists of those integers x which permit representations $x = a + b$, $a \subset A$, $b \subset B$. Let $A(n), \cdots$ denote the number of positive elements $\leq n$ of $A, \cdots$. Let $0 \subset B$, $1 \subset B$ and let $l(m)$ denote the smallest l for which $m = b_1 + \cdots + b_l$, $b_1 \subset B, \cdots, b_l \subset B$ is solvable. Put $\lambda = \sup_{k=1,\cdots,n} k^{-1} \sum_1^k l(m)$. Let $A \subseteq P \subseteq Q \subseteq C$. Generalizing his first paper [Math. Z. 62 (1955), 368–387; MR 17, 712], the author proves

(1) $\lambda k(C(n) - A(n)) - k(Q(n) - P(n)) \geq$

$$\begin{cases} \Sigma_1 - \dfrac{P(n)(P(n)-1)}{2} - P(n)(Q(n) - P(n)) + \Gamma(k) \\ k\bar{Q}(n) - \Sigma_2 - \dfrac{\bar{Q}(n)(\bar{Q}(n)-1)}{2} - \bar{Q}(n)(\bar{P}(n) - \bar{Q}(n)) + \bar{\Gamma}(k) \end{cases}$$

where $\Sigma_1 = \sum_{n-k}^{n-1} P(m)$, $\Sigma_2 = \sum_1^k \bar{Q}(m)$,

$$\Gamma(k) = \sum_1^{n-k-1} P(m)\{Q(m+k+1) - Q(m+k)\},$$

$$\bar{\Gamma}(k) = \sum_1^{n-k-1} \bar{P}(m)\{\bar{Q}(m+k+1) - \bar{Q}(m+k)\}.$$

The case $P = Q = A$ yields the author's previous main result. Specializing $P = A$, $Q = C$ and $P = Q = C$, the following estimates are obtained: Let $\lambda > 1$, $0 < \alpha = \inf_{k=1,\cdots,n} A(K)/k < 1$. Put

$$\gamma = \inf_{k=1,\cdots,n} \frac{C(k)}{k},$$

$$c(\varrho) = \frac{1 + \varrho^{\frac{1}{2}} + \varrho}{(1 + \varrho^{\frac{1}{2}})^2}, \quad c_3(\alpha, \lambda) = c(\alpha) \frac{\lambda}{\lambda - 1 + (1+\alpha)\alpha^{\frac{1}{2}}}.$$

Then

$$\frac{C(n)}{n} > \alpha(1 + c_3(\alpha, \lambda)(1-\alpha)\lambda^{-1}),$$

$$\frac{C(n)}{n} > \alpha + \gamma(1-\gamma) \cdot \lambda^{-1} \cdot \max(c(\gamma), c(1-\gamma)).$$

The author notes that $c_3(\alpha, \lambda) > 1$ if α is small enough. Similar results are derived from (1) for asymptotic densities, Finally, Erdös' original estimate

$$\frac{C(n)}{n} > \alpha\left(1 + \frac{(1-\alpha)}{2\lambda}\right)$$

is extended to sets of lattice points in n-space.

P. Scherk (Kingston, Ont.).

Barnes, E. S. The covering of space by spheres. Canad. J. Math. 8 (1956), 293–304.

Let $f(x) = f(x_1, \cdots, x_n)$ be a positive definite quadratic form of determinant D. Define the "inhomogeneous minimum"

$$m(f) = \sup_\alpha \inf_x f(x + \alpha),$$

where α and x run through all real vectors and all integer vectors respectively. The author, by analogy with the homogeneous case, says that $f(x)$ is extreme if the expression $D^{-1/n}m(f)$ is a (local) minimum, and absolutely extreme if it is a global minimum. For $n=3$ he shows that the only extreme forms are the multiples of

$$x_1^2 + x_2^2 + x_3^2 (x_1 - x_2)^2 + (x_2 - x_3)^2 + (x_3 - x_1)^2$$

and forms equivalent to them. He shows that these are absolutely extreme. It follows that

$$m(f) \geq (125D/1024)^{\frac{1}{3}}$$

whenever $n=3$, a result due to Bambah [Proc. Nat. Inst. Sci. India 20 (1954), 25–52; MR 15, 780]. The ingenious proof uses a type of reduction of quadratic forms developed by Voronoï [J. Reine Angew. Math. 133 (1908), 97–178; 134 (1908), 198–287; 136 (1909), 67–181]. It is claimed that this method is simpler than that of Bambah and opens up the possibility of attacking $n > 3$.

J. W. S. Cassels (Cambridge, England).

Davenport, H.; and Roth, K. F. Rational approximations to algebraic numbers. Mathematika 2 (1955), 160–167.

Let α be an irrational algebraic number, ζ be real and >0. Roth proved previously [Mathematika 2 (1955), 1–20, 168; MR 17, 242] Siegel's famous conjecture that $|\alpha - h/g| < q^{-2-\zeta}$ has only a finite number N of solutions in relatively prime rational integers h, g with $g > 0$. In the present paper there is obtained (theorem 1 and corollaries) an upper bound for N depending on ζ, the degree of α, and the maximum of the absolute values of the coefficients of the defining equation of α. Similarly, there is obtained (theorem 2) an upper bound for the number of diophantine solutions of $f(x, y) = g(x, y)$, where f is an irreducible form with rational integral coefficients, g is a polynomial with rational coefficients, and $\deg f - \deg g > 2$. Finally, it is shown (theorem 3) that if p_k/q_k is the kth

convergent to the continued fraction for a real algebraic number β then $\log\log q_k < c(\beta)k/\log k)^{\frac{1}{2}}$, where $c(\beta)$ is independent of k. *E. R. Kolchin* (Paris).

Chalk, J. H. H. Rational approximations in the complex plane. II. J. London Math. Soc. 31 (1956), 216–221. [For part I see same J. 30 (1955), 327–343; MR 17, 17.]

The author proves the theorem of Hlawka, (accounting for extreme values), [Monatsh. Math. 46 (1938), 324–334], that for any complex numbers u_0, v_0 there exists gaussian integers u, v satisfying

$$|f(u+u_0, v+v_0)| \leq \tfrac{1}{3}|\Delta|,$$

where $f(u, v) = (\alpha u+\beta v)(\gamma u+\delta v)$, and $\alpha, \beta, \gamma, \delta$ are complex with nonvanishing determinant Δ. The theorem is referred to the Ford-like configuration of spheres $S(\Omega, R)$ of radius R lying in the upper half ξ, η, ζ space, tangent to plane $\zeta=0$, and with center directly above $\Omega=\xi+i\eta$. Thus if $\varrho e^{i\theta}$ $(\varrho^2\geq 2)$, u_0' are given complex constants, then for some gaussian integer u' the sphere $S(\varrho e^{i\theta}(u'+u_0'), \frac{1}{2}\varrho^2)$ will intersect a preassigned circle orthogonal to the ζ-plane. *Harvey Cohn* (Washington, D.C.).

Sós, Vera T.; and Turán, P. On some new theorems in the theory of Diophantine approximations. Acta Math. Acad. Sci. Hungar. 6 (1955), 241–255. (Russian summary)

The principal result of this paper is the following improvement of the fundamental Theorem II of the author's recent book, Über eine neue Methode in der Analysis und deren Anwendungen [Akad. Kiadó, Budapest, 1953; MR 15, 688]. Let m be a non-negative integer, and suppose that the b_j's and z_j's are complex numbers with $|z_1|\geq\cdots\geq|z_n|$. Let A_1 be the infimum of the positive real numbers α such that there exists an integer ν with $m+1\leq\nu\leq m+n$ for which

$$(*)\qquad |b_1z_1^\nu+\cdots+b_nz_n^\nu| \geq |z_1|^\nu\left(\frac{n}{\alpha(n+n)}\right)^n\cdot \min_{1\leq j\leq n}(|b_1+\cdots+b_j|).$$

Then $1.321 < A_1 < 2e^{1+4/e}$ (<24) if $b_1=\cdots=b_n=1$. A similar proof shows that the inequality (*) holds for some ν in the specified range and for arbitrary $b_1, \cdots, b_n$, if $\alpha=2e^{1+4/e}$. (The earlier value of α was $24e^2\approx 177$.) In the latter part of the paper special systems $z_1, \cdots, z_n$ are considered, for which $s_2=s_4=\cdots=s_{n+1}=0$, where $s_\nu = z_1^\nu+\cdots+z_n^\nu$. It is shown that all such systems are given by the sets of zeros of the polynomials

$$f_n(z, a, \lambda) = z^n + \frac{H_1(\lambda)}{1!}az^{n-1}+\cdots+\frac{H_n(\lambda)}{n!}a^n,$$

where H_ν is the νth Hermite polynomial, λ is any zero of H_{n+1} and a is an arbitrary complex number.

W. J. LeVeque (Ann Arbor, Mich.).

See also: Walsh, p. 1077.

ANALYSIS

Have

★**Rothe, R.; und Schmeidler, W. Höhere Mathematik für Mathematiker, Physiker, Ingenieure. Teil VII. Räumliche und ebene Potentialfunktionen. Konforme Abbildung. Integralgleichungen. Variationsrechnung.** B. G. Teubner Verlagsgesellschaft, Stuttgart, 1956. 218 pp. DM 22.40.

The subject matter of this volume is presented in three chapters, the first of which is devoted to harmonic functions in space, and harmonic and analytic functions in the plane. In addition to discussion of the Newtonian and logarithmic potentials there are sections on conformal mapping, the reduction of elliptic integrals, and elliptic functions; an appendix to Chapter I deals with the two-dimensional equation $\Delta\Delta u=0$ and related problems. Chapter II is concerned with systems of linear algebraic equations and integral equations of the second and first kind. In Chapter III on the calculus of variations there is a discussion of simple problems in non-parametric and parametric form, variational problems of differential geometry, the Weierstrass theory, and the Hamilton-Jacobi theory. The elementary discussion of the text, together with the accompanying problems and references to literature, make this little book a good introduction to the disciplines under discussion.

W. T. Reid.

Williamson, R. E. Multiply monotone functions and their Laplace transforms. Duke Math. J. 23 (1956), 189–207.

A real $f(t)$ $(t>0)$ is n times monotone for integral $n\geq 2$ (written $f\in K_n$) if $(-1)^kf^{(k)}(t)$is non-negative, non-increasing and convex for $k=0, 1, \cdots, n-2$, $t>0$. For a completely monotone function $(-1)^kf^{(k)}(t)\geq 0$, $t>0$, $k=0, 1, \cdots$.

It is shown that $f\in K_n$ if and only if

$$f(t)=\int_0^\infty (1-ut)_+^{n-1}d\gamma(u)$$

where $g_+=\frac{1}{2}(|g|+g)$ and $\gamma(u)$ is non-decreasing and bounded below. If $\gamma(0)=0$ then $\gamma(u)$ is unique at its points of continuity and

$$\gamma(u)=\frac{(-1)^{n-1}}{(n-1)!}\int_{0+}^u x^{-n+1}df^{(n-1)}\left(\frac{1}{x}\right)+f(\infty)= \sum_{k=0}^{n-1}\frac{(-1)^kf^{(k)}(1/u)}{k!}\left(\frac{1}{u}\right)^k.$$

Further, if $f\in K_n$, $g\in K_n$, then $fg\in K_n$.

For arbitrary $\alpha>1$ (α not necessarily integral) f is defined to be α times monotone if $f(t)=\int_0^\infty(1-ut)_+^{\alpha-1}d\gamma(u)$, $\gamma(u)$ non-decreasing and bounded below. Then $f\in K_\alpha$ if and only if (i) $D^{\alpha-2}[t^{\alpha-1}f(1/t)]$ is non-negative, non-decreasing and convex for $t>0$ and (ii) $\lim_{t\to\infty} f(t)$ exists and is non-negative. Here

$$D^\beta f = D^nI^{n-\beta}f = D^n[\Gamma(n-\beta)]^{-1}\textstyle\int_0^t(t-u)^{\beta-1}f(n)du.$$

It is also shown that if $1\leq\beta<\alpha$, then $K_\beta\supset K_\alpha$.

A function $f(s)$ $(0<s<\infty)$ is shown to be the Laplace transform of $\varphi\in K_\alpha$ where φ is summable in every finite interval if and only if (i) $D^\alpha[s^\alpha f(s)]$ is completely monotone and summable at the origin, (ii) $\lim_{s\to 0} f(s)=0$, (iii) $\lim_{s\to\infty} sf(s)$ exists.

Finally, if $\varphi\in K_3$ and is summable in every finite interval, the Laplace transform of φ is univalent in Re $(s)>0$. *F. Goodspeed* (Vancouver, B.C.).

Ionescu, D. V. From Archimedes' formula to a cubature formula. Gaz. Mat. Fiz. Ser. A. 8 (1956), 3–10. (Romanian)

The remainder term in Simpson's formula for numerical

integration vanishes, provided that the integrand is a polynomial of degree 3, at most. Archimedes' formula, giving the area of a parabolic segment is obtained as a particular case. Furthermore, Simpson's formula is used to compute volumes, whose sections by parallel planes are polynomials of at most third degree of the ordinate. Among the less trivial results is the following: Let $C(z_0)=S(z_0)(b-a)$ be the volume of the cylinder of height $b-a$ and basis $S(z_0)$, where $S(z_0)$ is the area of the section of some solid K, by the plane $z=z_0$; let V_s be the volume of the sphere tangent to the planes $z=a$ and $z=b$; finally, let $V_a{}^b$ be the volume of K between $z=a$ and $z=b$. Then, if K is the cone $x^2+y^2-z^2=0$, $V_a{}^b=C(\frac{1}{2}(a+b))+\frac{1}{2}V_s$, while if K is the sphere $x^2+y^2+z^2=R^2$, $V_a{}^b=C(\frac{1}{2}(ab))-\frac{1}{2}V_s$. *E. Grosswald* (Philadelphia, Pa.).

Rubio Vidal, Francisco Javier. An arithmetic progression of infinite order. Las Ciencias 19 (1954), 293–296. (Spanish)

The author introduces the sequence R_n: 1, 2, 5, 15, $\cdots$, defined by $\sum_0^\infty (k+1)^{n-1}/k!=R_n\cdot e$. He notes that $(\Delta^{(n)}R_n)_1=R_n$ and has thus a simple example of an arithmetic progression of infinite order. The expansion

$$e^{e^x}=e\left[1+\sum_{n=1}^{n=\infty}\frac{R_n}{n!}x^n\right]$$

gives an interesting application. *H. W. Brinkmann.*

Theory of Sets, Functions of Real Variables

Have

★ **Bourbaki, N. Eléments de mathématique. XX. Première partie: Les structures fondamentales de l'analyse. Livre I: Théorie des ensembles. Chapitre III: Ensembles ordonnés. Cardinaux. Nombres entiers.** Actualités Sci. Ind., no. 1243. Hermann & Cie, Paris, 1956. ii+118 pp.

The first volume of this book [Eléments de mathématique, XVII, Hermann, Paris, 1954; MR 16, 454] was devoted mainly to the logical machinery needed in axiomatic set theory; the present volume is concerned with set theory itself. The contents are roughly those of an introductory course on cardinal numbers. Except in occasional minor detail the treatment is standard. It begins with a discussion of order and the usual circle of ideas clustering around that concept (e.g., directed sets and direct and inverse limits). Well-ordering is introduced and the relation between Zermelo's theorem and Zorn's lemma is discussed. Ordinal numbers are treated rather cavalierly (they appear in the exercises only), but elementary cardinal arithmetic (through the fact that every infinite cardinal is idempotent) receives a more leisurely treatment. (In view of the authors' feelings about ordinals, they are barred from using von Neumann's elegant definition of cardinals; the spirit of the definition they adopt is that of Cantor's.) Natural numbers are defined as finite cardinals, and the definition is followed by a treatment of elementary number theory (through the Euclidean algorithm and the basic facts about combinatorics). *P. R. Halmos* (Chicago, Ill.).

Doss, Raouf. On Riemann integrability and almost periodic functions. Compositio Math. 12 (1956), 271–283.

Let $f(x)$ be a complex function of a real variable with the property that to every $\varepsilon>0$ there exist two trigonometric polynomials $p(x)$, $q(x)$ such that $p(x)\ll f(x)\ll q(x)$ (where $a\ll b$ means Re $a\leq$Re b and Im $a\leq$Im b) and such that the Weyl-distance, or Besicovitch distance, between $p(x)$ and $q(x)$, i.e., $M\{q(x)-p(x)\}$, is $<\varepsilon$. Such a function is called a R.W.a.p. or R.B.a.p. function (R for Riemann). The functions of this type are characterized in various ways by structural properties; one of these ways has previously been announced by the author [C.R. Acad. Sci. Paris 238 (1954), 317–318; MR 15, 620]. Also the analogous notion of a R.S.a.p. function (S for Stepanoff) is studied.

E. Følner (Copenhagen).

Hartman, S. Über die Verteilung der Fastperioden von fastperiodischen Funktionen auf Gruppen. Studia Math. 15 (1955), 56–61.

After some statements, which hold in the general, non-abelian case, the main part of the paper deals with the set of ε-translation elements of an almost periodic function $f(x)$ on an abelian group G, for fixed $\varepsilon>0$. An element $\tau\in G$ is called an ε-translation element if $|f(x+\tau)-f(x)|<\varepsilon$ for all $x\in G$.

Let $\sum_{\nu=1}^\infty a_\nu\chi_\nu(x)$ be the Fourier series of $f(x)$. If $s\in G$ is chosen so that $\chi_1^{r_1}(s)\cdots\chi_m^{r_m}(s)$ for any m and any integers $r_1, \cdots, r_m$ is equal to 1 only when $\chi_1^{r_1}\cdots\chi_m^{r_m}$ is the principal character, then for all $\varepsilon>0$, with at most a denumerable number of exceptions, the ε-translation elements in the arithmetical progression $t+ns$ (t fixed $\in G$, $n=0, \pm1, \pm2, \cdots$) occur with a positive frequency, which does not depend on t and s, but only on ε. — In the proof the Bohr compactification of G by $f(x)$ is an essential tool. *B. Jessen* (Copenhagen).

Ichijô, Yoshihiro. Über die Laplacesche asymptotische Formel für das Integral von Potenze mit grossem Indexe. J. Gakugei Tokushima Univ. Nat. Sci. Math. 6 (1955), 63–74.

Let $\varphi(x)$ and $f(x)$, $x=(x^1, \cdots, x^m)$, be defined in an interval B of euclidean m-space, and satisfy the following conditions: (1) $f(x)\geq0$ and $\varphi(x)[f(x)]^n$ is integrable in B for $n=1, 2, \cdots$; (2) $F(x)=\log f(x)$ is regular in B and attains its maximum in B at a single interior point ξ; (3) $\varphi(x)$ is regular in a neighborhood of ξ and $\varphi(\xi)\neq0$. Then as $n\to\infty$ the following asymptotic formula holds:

$$(*)\int\cdots_B\int\varphi f^n dx^1\cdots dx^m\cong\frac{(2\pi/n)^{m/2}}{((-1)^mD)^{\frac{1}{2}}}[f(\xi)]^n\left\{\varphi(\xi)+O\left(\frac{1}{n}\right)\right\},$$

where D stands for the determinant $|\partial^2F/\partial x^i\partial x^j|_{x=\xi}$. For $m=1$, this reduces to a formula of Laplace [Pólya and Szegö, Aufgaben und Lehrsätze aus der Analysis, vol. 1, Springer Berlin, 1925, pp. 78, 244 . The author also obtains results for the case when F takes its maximum on a manifold than rather at a single point. Finally, for $m=1, 2$ he gives a more refined estimate, where in (*) $O(1/n)$ is replaced by $n^{-1}\psi(\xi)+O(1/n^2)$ with ψ a certain complicated function involving the derivatives of F up to and including order 4.

W. H. Fleming.

Jacobsthal, Ernst. Beiträge zur Differentialrechnung. II. Norske Vid. Selsk. Forh., Trondheim 28 (1955), 42–45.

In part I [same Forh. 27 (1954), 146–150; MR 17, 21] the author has developed a formula for the nth derivative of $F(x)=g(x)/f(x)$.

This formula is

$$f^{n+1}\cdot F^{(n)}=\sum_{\lambda=0}^{n}(-1)^{n}f^{n-\lambda}(f^{\lambda}\cdot g)_0^{(n)}.$$

The symbol $(f^{\lambda}g)_0^{(n)}$ represents the aggregate of all the terms of the nth derivative of $f^{\lambda}\cdot g$ in which the function f does not occur. The present note uses the method of induction to give a shorter and more easily followed proof of this formula. *R. L. Jeffery* (Kingston, Ont.).

Thale, James S. Univalence of continued fractions and Stieltjes transforms. Proc. Amer. Math. Soc. 7 (1956), 232–244.

It is shown that, if ϕ is a nondecreasing function on the interval $[0, 1]$ which is not constant on $(0, 1]$, then the functions $F(x)=\int_0^1[1+xt]^{-1}d\phi(t)$ and $xF(x)$ are univalent on the set M: $\operatorname{Re} x\geqq -1$, $x\neq -1$; and there exist such functions not univalent on a set containing M as proper subset. The functions F are just those nonconstant functions having continued fraction expansions of the form $c/1+(1-g_1)g_2x/1+(1-g_2)g_3x/1+\cdots, c>0, 0\leqq g_p\leqq 1$. The author considers functions having other continued fraction expansions. If $F(x)=1/1+a_1x/1+a_2x/1+\cdots$, $|a_p|\leqq\frac{1}{4}$, $a_1\neq 0$, then F is univalent for $|x|<4[3\sqrt{(2)}-4]=.968\cdots$; if $F(z)=1/(b_1+z)-a_1^2/(b_2+z)-\cdots$, a positive definite continued fraction with $|a_p|<M/3$, then F is univalent for $\operatorname{Im} z>M\sqrt{(2)}/3$; and if $F(z)=1/(b_1+z)-a_1^2/(b_2+z)-\cdots$, $|a_p|<N/3$, $|b_p|<M/3$, then F is univalent for $|z|>(N/\sqrt{(2)})+M/3$. *H. S. Wall.*

Motchane, Léon. Sur un nouveau critère de conservation de classe de Baire. C. R. Acad. Sci. Paris **242** (1956), 605–608.

$\{f_k\}$ désigne une suite de fonctions numériques définies sur l'intervalle $\mathfrak{J}_0$ des nombres irrationnales entre 0 et 1, de classe α, convergeant ponctuellement vers une fonction f. Utilisant la terminologie et les résultats de son mémoire au J. Math. Pures Appl. (9) **34** (1955), 337–394 [MR **17**, 650], l'auteur, s'inspirant de C. Kuratowski, envisage une fonction de classe α comme une fonction associé à un système essentiel de classe α additive et énonce le théorème suivant: (T) Pour que f soit de classe α, il faut et il suffit que la famille $\{f_k\}$ soit équi-associée à un système essentiel $\mathfrak{E}$ de classe α additive. En vue d'obtenir $\mathfrak{E}$ est introduit le procédé suivant dit „régulier": $\mathfrak{E}_1$ est un système essentiel associé à f_1. $\mathfrak{E}_1, \cdots, \mathfrak{E}_K$ $(K\geqq 1)$ étant déjà définis, on passe de $\mathfrak{E}_K$ à $\mathfrak{E}_{K+1}$ en subdivisant tout ensemble de $\mathfrak{E}_{K,n}$ $(n=1, 2, \cdots)$ sur lequel l'oscillation de f_{K+1} est $>1/n$ en sous-ensembles de classe α sur lesquels l'oscillation de f_{K+1} est $\leqq 1/n$. La possibilité d'obtenir de tels schémas de Souslin $\mathfrak{S}_n$ tous dénombrables équivaut à l'appartenance de f à la classe α; le système $\mathfrak{E}$ des intersections des ensembles des schémas satisfait alors aux conditions de (T). Une méthode de construction d'une fonction de classe stricte $\alpha+1$ est déduite du critère précédent. Les résultats de la note restent valables si les fonctions f_k sont définies sur un espace métrique X et prennent leurs valeurs dans un espace métrique Y séparable. [Remarque du rapporteur: Le procédé „régulier" est défini plus haut ainsi que le rapporteur croit l'avoir compris. Il lui semble que chaque $\mathfrak{S}_n$ est défini comme disjoint dans le cas où $X=\mathfrak{J}_0$, Y=droite numérique.] *C. Pauc* (Nantes).

Császár, Ákos. Sur les fonctions localement monotones au sens généralisé. Acta Math. Acad. Sci. Hungar. 6 (1955), 451–461. (Russian summary)

Let $\mathfrak{N}$ be an hereditary σ-additive family of sets in R^1: U be the set of points x_0 such that, for some $\delta>0$, neither of the intervals $(x_0-\delta, x_0)$, $(x_0, x_0+\delta)$ belongs to $\mathfrak{N}$. A real function $f(x)$ is said to have the property C^* at x_0 (that is, to be locally increasing in a generalized sense) if, for some $\delta>0$, the set

$$\left\{\frac{f(x-f(x_0)}{x-x_0}<0,\ 0<|x-x_0|<\delta\right\}$$

belongs to $\mathfrak{N}$: similarly the property D^* is defined by replacing the first $<$ in the above definition by $>$: finally the function $f(x)$ has the property M^* at x_0 if it has property C^* or property D^*. Theorems are proved concerning functions which have this property of being locally monotone when sets of the family $\mathfrak{N}$ are ignored, or similar properties. For example: If $f(x)$ has property M^* at all points of a set $S\subset U$ then (1) $S=\bigcup_{n=1}^{\infty}S_n$, where $S_n\subset S_{n+1}$ and $f(x)$ is monotone (in the ordinary sense) by segments on each S_n; (2) $f(x)$ belongs on S to the second Baire class at most; (3) $f(x)$ is continuous (in a generalised sense ignoring sets in $\mathfrak{N}$) at all points of S except a countable set. *U. S. Haslam-Jones* (Oxford).

Jurkat, W. B. An extension problem for functions with monotonic derivatives. Canad. J. Math. 8 (1956), 184–191.

Problem (A): Let $F(x)$ be the nth integral of a positive nondecreasing function for large positive x; can $F(x)$ be extended to a function $f(x)$ which vanishes near $-\infty$ and is the nth integral of a nondecreasing function for all x? The author reduces this to a boundary value problem for functions with monotonic derivatives and solves the latter by showing its equivalence with a reduced moment problem whose solubility depends on the positive definite character of certain quadratic forms. For logarithmico-exponential functions (in the sense of Hardy) the character of the quadratic forms can always be determined: for such functions, problem (A) has a solution for $n\geqq 2$, if and only if $x^{-n}F(x)$ is nondecreasing for large positive x; for $n=0$, 1 there is always a solution.

R. P. Boas, Jr. (Evanston, Ill.).

Haršiladze, F. I. On the modulus of continuity of certain classes of functions. Leningrad. Gos. Univ. Uč. Zap. **137.** Ser. Mat. Nauk **19** (1950), 155–159. (Russian)

The author gives a short and elementary proof of the following result (Lemma 4) of Zygmund [Duke Math. J. **12** (1945), 47–76; MR **7**, 60]: If f is continuous and periodic and satisfies $|f(x+h)+f(x-h)-2f(x)|=O(h^{\alpha})$ uniformly in x and $h>0$ for a fixed $0<\alpha\leqq 1$, then $f\in \operatorname{Lip}\alpha$ if $\alpha<1$ and has modulus of continuity $\omega(h)=O(h|\log h|)$ if $\alpha=1$. (Zygmund further showed that the hypothesis concerning the continuity of f can be replaced by the assumption that f is measurable). Zygmund's proof of the above result, though elegant, requires that one know something about the theory of best (trigonometric) approximation; the present proof requires only a minimum knowledge of the calculus.

The author first establishes the estimate

$$\omega(h)\leqq c[\delta^{\alpha}+h\delta^{-1}\omega(\delta)]$$

by introducing the function $f_{\delta}(x)=(2\delta)^{-1}\int_{x-\delta}^{x+\delta}f(t)dt$, $\delta>0$, considering the difference $f_{\delta}(x)-f(x)$, and applying the mean-value theorem to $f_{\delta}(x+h)-f_{\delta}(x)$. He then verifies by induction that $\omega(h)\leqq c(1-h^{1-\alpha})h^{(n-1)\alpha/n}/[1-h^{(1-\alpha)/n}]$ $(\alpha<1)$ and $\omega(h)\leqq cnh^{(n-1)/n}$ $(\alpha=1)$ for $n=2, 3, 4, \cdots$ (The case $n=2$ is obtained from his first estimate above by

setting $\delta=h^{\frac{1}{2}}$). The proof is closed by choosing

$$n\leqq-\log h<n+1$$

for h small. *A. E. Livingston* (Seattle, Wash.).

Motchane, Léon. Sur la construction effective de fonctions de classe quelconque. Bull. Sci. Math. (2) **79** (1955), 180–190.

Quant à la terminologie dans C. R. Acad. Sci. Paris **233** (1951), 1569–1571 [MR **13**, 857], l'A. reprend le problème de l'existence des fonctions de Baire de chaque classe. Il fait voir comment la construction de fonctions de classes de plus en plus élevées peut être basée sur quelques notions générales que l'A. avait définies antérieurement [loc. cit.].

Théorème A: Soit f_k une suite de fonctions de classes $\leqq\alpha$; pour que $\lim f_k$ soit de classe $\leqq\alpha$, il faut et il suffit que la famille $\{f_k\}$ soit „équi-associée à un système essentiel de la classe additive". L'A. donne deux autres énoncés (th. B, C) pour assurer que $\lim f_k$ soit de classe $\leqq\alpha$ et ayant des liens avec ce qu'a fait L. Keldych [C.R. (Dokl.) Acad. Sci. URSS (N.S.) **26** (1940), 523–525; **28** (1940), 675–677; **31** (1941), 651–653; MR **2**, 256; **3**, 226].

G. Kurepa (Zagreb).

Garder, Arthur O., Jr. The zeros of quasi-analytic functions. Proc. Amer. Math. Soc. **6** (1955), 929–941.

Soit $\nu(x)$ une fonction possédant une dérivée continue pour $x\geqq0$, avec $\nu(0)=1$, $\nu'(x)\geqq0$, $x\nu'(x)/\nu(x)=o(\log x)^{-1}$ $(x\to\infty)$, et soit $M_n^*=n![\nu(n)]^n$. On dit que

$$f(x)\in C(M_{n,k}^*)=C^*\ (-\infty<x<\infty)$$

si $|f^{(n)}(x)|\leqq Ak^nM_n$ $(n\geqq0)$. Soit $Z(u)$ une fonction non supérieure au nombre des zéros de $f(x)$ sur l'intervalle $|x|\leqq u$. L'auteur donne une nouvelle démonstration du théorème suivant de Hirschman [Amer. J. Math. **72** (1950), 396–406; MR **11**, 583]. Posons

$$T(u)=\max_{n\geqq0}u^n(M_n^*)^{-1},\quad H(v)=2\pi^{-1}\textstyle\int_1^v u^{-2}\log T(u)\,du.$$

Si $H(\infty)=\infty$ et s'il existe une fonction $Z(u)$ telle que $\limsup_{u\to\infty}H[Z(u)]/u>k$, alors $f(x)\equiv0$.

S. Mandelbrojt (Paris).

Gehér, László. On approximately differentiable functions of two variables. Acta Math. Acad. Sci. Hungar. **6** (1955), 439–444. (Russian summary)

The author rediscovers a known result. The reviewer remarked years ago [Fund. Math. **35** (1948), 275–302, p. 299, th. (9–3); MR **10**, 520] that it is implicitly contained in Saks, Theory of the integral [2nd ed., Warsaw, 1937, p. 300]. *L. C. Young* (Madison, Wis.).

Bakel'man, I. Ya.; and Verner, A. L. Generalized derivatives of continuous functions of two variables. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1(67), 173–179. (Russian)

Let f be a continuous function defined in the (x, y) plane. The authors show that each of the following conditions, (I) and (II) below, is equivalent to the existence of a pair of functions which are partial derivatives of f in the sense of Sobolev or Laurent Schwartz: (I) the function of sets defined by the plane variation of f is absolutely continuous; (II) there exist summable functions P, Q such that, for each positive integer n and each pair of points A, B of the plane, the difference $f(B)-f(A)$ is expressible as the integral of $Pdx+Qdy$ along almost every n-sided polygon, with sides parallel to the axes which joins A to B.

L. C. Young (Madison, Wis.).

Marcus, S. Sur la détermination d'une fonction partiellement continue par des valeurs prises sur un ensemble dense. Com. Acad. R. P. Romîne **5** (1955), 1563–1568. (Romanian. Russian and French summaries)

Let E be a dense subset of an open set G in Euclidean n-space and let $f(x_1, \cdots, x_n)$ be a real function which vanishes in E and is continuous in each variable separately at each point of $G-E$. The author proves that f vanishes identically in G. *L. C. Young* (Madison, Wis.).

Jûza, Miloslav. Sur les fonctions non dérivables. Czechoslovak Math. J. **5(80)** (1955), 371–381. (Russian. French summary)

A sequence $\{G_n(t)\}$ of real functions is constructed with the following property: if $P(x_1, \cdots, x_p)$ is any real function possessing, at every point $(x_1, \cdots, x_p)$, partial derivatives of any order at least one of which is different from zero, then $P(G_1(t), \cdots, G_p(t))$ is continuous and nondifferentiable at every point t. *F. A. Behrend.*

de Groot, J. A system of continuous, mutually nondifferentiable functions. Math. Z. **64** (1956), 192–194.

A set S, of power aleph, of continuous real functions is constructed such that for any two functions $F, G\in S$, F is nowhere differentiable with respect to G (i.e. the lower and upper limits, as $h\to0$, of $(F(x+h)-F(x))/(G(x+h)-G(x))$ are different for all real x). *F. A. Behrend.*

Ridder, J. Die Einführung von beschränkt- und totaladditivem Mass. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 143–154, 155–165.

I. Given are a set R, a ring K of subsets of R and a bounded additive function $\Phi(P)$ on K. Defined are the ring K_1 of sets in R for which the outer and inner measures T_a, T_i of the total variation T of Φ have meaning, the ring $K_2\subset K_1$ of Φ measurable sets P for which $T_a(P)=T_i(P)$, and the ring K_3 of Φ measurable sets P, in the extended sense, for which $P\cdot M\in K_2$ for every $M\in K$. The measure m_Φ, extension of Φ, is finite on K_2 but may have infinite values on K_3. For a pair of sets $R^{(1)}$, $R^{(2)}$ and associated rings $K^{(1)}$, $K^{(2)}$ and bounded additive functions $\Phi^{(1)}$, $\Phi^{(2)}$, the product measure is defined by first letting K be the smallest ring containing all sets $P\times Q$, $P\in K^{(1)}$, $Q\in K^{(2)}$ and defining Φ on K in the natural way, and then proceeding as above. A similar development is made for completely additive measures. The pathology of the non σ finite for product measures is avoided by the device of defining the measure first for K_2 and then making the extension. [Cf. P. R. Halmos, Measure theory, Van Nostrand, New York, 1950, Chs. 2 and 7; MR **11**, 504; N. G. de Bruijn and A. C. Zaanen, Nederl. Akad. Wetensch. Proc. Ser. A. **57** (1954), 456–466; MR **16**, 228].

II. The considerations of I are used in defining integrals as measures of ordinate sets in suitable product spaces.

G. Goffman (Norman, Okla.).

Gonin, E. G. A new variant of the theory of nonnegative real numbers. Molotov. Gos. Ped. Inst. Uč. Zap. **1954**, no. 13, 47–52. (Russian)

Letting B be any infinite set of natural numbers >0, the author considers the set F of all natural-number-valued functions α defined on B satisfying the condition $[n\alpha(p)/p]\leqq\alpha(n)$ $(n, p\in B)$, and obtains the set of real

numbers $\geqq 0$ as the set of equivalence classes of a certain equivalence on F. *E. R. Kolchin* (Paris).

Busulini, Bruno. La relazione mediata nella matematica. Ann. Univ. Ferrara. Sez. VII. (N.S.) 4 (1954–1955), 69–80.

Generalities on relations in mathematics. Some examples from U. Morin, Algebra astratta [Part I, Milani, Padova, 1955] are examined to illustrate the difference between constitutive or fundamental relations and other link relations acting outside of the fundamental domain. The importance of analogy is stressed. *G. Kurepa.*

Busulini, Bruno. Relazione ad intra e relazione ad extra nella matematica. Ann. Univ. Ferrara. Sez. VII (N.S.) 4 (1954–1955), 91–106.

Sequel to the paper reviewed above. To the distinction of induction and deduction is associated the distinction of "relations ad intra" and "relations ad extra". Induction and deduction are not isolated one from another. At the same time when we do inductions, we are doing deduction too. *G. Kurepa* (Zagreb).

Kurepa, G. Some principles of induction. Acad. Serbe Sci. Publ. Inst. Math. 8 (1955), 1–12.

Contents of lectures delivered at Wayne University and Purdue University, Oct. and Nov., 1950. Four induction principles are given: (1) for any set, (2) for any (partially) ordered set S, (3) for certain ranged sets, (4) for certain ramified sets (S is ranged if every nonempty subset has a minimal element, ramified if the set of all predecessors of each element is totally ordered). The condition in (3) appears as follows; if every element is comparable with a minimal element, then S is ranged if and only if the following induction principle holds: if M contains $x \in S$ whenever M contains every predecessor of x, then $M \supseteq S$. The condition in (4) appears as follows; if S is a ramified set, then $A_4(S)$ is equivalent to $B_4(S)$, where $A_4(S)$ means that every maximal chain in S is without gaps, and $B_4(S)$ is the stated induction principle (not quoted here). Examples are given to show that, without the ramification condition, the properties $A_4(S)$, $B_4(S)$ are independent. *L. Gillman* (Lafayette, Ind.).

Locher-Ernst, L. Merkwürdiges vom Kontinuum. Elem. Math. 11 (1956), 49–50.

Discussion of the fact that the countable set of rationals has uncountably many gaps, while its uncountable complementary set has only countably many gaps. *L. Gillman* (Lafayette, Ind.).

Kapuano, Isaac. Questions apparentées au problème du continu. C. R. Acad. Sci. Paris 242 (1956), 1833–1836.

Décomposition de tout ensemble analytique ou complémentaire analytique non dénombrable en $\aleph_1$ ensembles boréliens deux à deux disjoints de classes $\leqq 3$. Relations avec les ensembles de Denjoy des ensembles toujours de première catégorie et des complémentaires analytiques. (Author's summary.) *L. Gillman* (Lafayette, Ind.).

Kapuano, Isaac. Sur un problème de Lusin concernant la décomposition du continu linéaire. C. R. Acad. Sci. Paris 242 (1956), 978–981.

Tout ensemble linéaire M jouissant de la propriété de Baire est réunion d'un ensemble toujours de première catégorie et de $\alpha \leqq \aleph_1$ ensembles boréliens de classes $\leqq 3$ et deux à deux disjoints. La proposition est déduite des propriétés des continus héréditairement indécomposables. (Author's summary).

L. Gillman (Lafayette, Ind.).

Yavec, M. A. On separability of Borel elements in a nonlinear partially ordered space. Leningrad. Gos. Univ. Uč. Zap. 137. Ser. Mat. Nauk 19 (1950), 53–58. (Russian)

This note discusses an axiomatization of the partial order properties of the family of Borel sets. In this axiom scheme it is proved that if $y \leqq x$ and if y is in the αth intersection class and x in the αth union class there is an element z in both classes with $y \leqq z \leqq x$.

M. M. Day (Urbana, Ill.).

Mardešić, Sibe. Powers of intersections between Jordan curves and straight lines of the plane. Hrvatsko Prirod. Društvo. Glasnik Mat.-Fiz. Astr. Ser. II. 10 (1955), 137–160. (Serbo-Croatian summary)

Let S denote the set whose elements are $2^{\aleph_0}$, $\aleph_0$, and the nonnegative integers, let K be any subset of S that contains 0, 1, and $2^{\aleph_0}$, and and let L be the set of all straight lines in the plane. Then there exists a Jordan curve J in the plane, such that $\{|J \cap l| : l \in L\} = K$.

F. Bagemihl (Notre Dame, Ind.).

Shirai, Tamesharu. A remark on the ranged space. Proc. Japan Acad. 32 (1956), 120–124.

Instead of the C-axiom of Hausdorff (for every point p and every neighbourhood $V(p)$ of p, if $q \in V(p)$, there is a $V(q) \subseteq V(p)$) the weaker axiom C' is considered: There exists a $q \in V(p)$ such that $V(q) \subseteq V(p)$.

The kernel $K[V(p)]$ of $V(p)$ is defined as the set of all such points $q \in V(p)$. The topology T^* is defined by means of kernels as neighbourhoods. This topology is equivalent with the closure topology $\hat{A}$ defined in the following way: A given, $\hat{A}$ consists of all points p such that for each $V(p)$ there exists a point $a \in A$ and a $V(a)$ such that $V(a) \subseteq V(p)$. Of course, $\hat{A} \subseteq \bar{A}$, $\hat{\hat{A}} \subseteq \hat{A}$. The topology T^* is strictly stronger than the topology T defined by means of $V(p)$'s. A modification of Baire's category theorem is given.

G. Kurepa (Zagreb).

Kurepa, Georges. Ensembles ordonnés et leurs sous-ensembles bien ordonnés. C. R. Acad. Sci. Paris 242 (1956), 2202–2203.

Three results are stated, the first being incorrect. [Counterexample: Let E be the negative integers. For each subset S of E let $f(S) = \max\{x/x \text{ in } E\} - 1$]. Let E be a partially ordered set and AE the smallest cardinal number equal to or greater than the power of any family of disjoint antichains [=sets of incomparable elements] of E whose union is E. Suppose that $AE \leqq \aleph_\sigma$, $\aleph_\sigma$ being regular. Then there exists a mapping f from E to H_σ, the order type of H_σ being η_σ, such that $f(a) < f(b)$ in H_σ whenever $a < b$ in E. Finally let σQ be the well ordered subsets of the rational numbers, ordered as follows: $M < (i) N$ if M is an initial segment of N. Then there does not exist any denumerable family of antichains of σQ whose union exhaust σQ.

S. Ginsburg (Hawthorne, Calif.).

See also: Hiraguti, p. 1045.

Have

Theory of Measure and Integration

★ **Haupt, Otto; Aumann, Georg; und Pauc, Christian Y. Differential- und Integralrechnung unter besonderer Berücksichtigung neuerer Ergebnisse. Bd. III. Integralrechnung. 2te Aufl.** Walter de Gruyter & Co., Berlin, 1955. xi+320 pp. DM 28.00.

This third volume of the new edition is a thoroughly modern textbook on measure theory and integration. The authors strive always for the greatest generality and adopt where possible the abstract viewpoint. The five parts, which include a total of eleven chapters, deal with, first, content, measure, and their extensions; second, partition integrals, σ-additive functions, and linear functionals; third, measure and integration in topological spaces; fourth, primitive functions and the indefinite integral; and fifth, applications.

A few of the many topics included are: lattices, σ-lattices, Boolean lattices, vector lattices, Boolean ideals, filter bases, filters, ultrafilters, Moore-Smith convergence, content, Jordan measure, Borel measure, Lebesgue measure, functions of Baire, step functions, convergence in measure, convergence in the mean, product measures, Fubini's theorem, Fatou's lemma, Vitali's lemma, generalized Riemann integrals, Stieltjes integrals, the Birkhill integral, the Denjoy integral, absolute continuity, additive set functions, functions of intervals, derivatives of set functions, Hilbert spaces, area, k-dimensional measures in n-space, mean value theorems, and the theorems of Green, Gauss and Stokes. There is a good index, and almost all the references in the bibliography are from the last ten years. There are no integral tables, and the engineering student will not learn from this book how to integrate the elementary functions. *O. Frink.*

Denjoy, Arnaud. La genèse des métriques boréliennes. C. R. Acad. Sci. Paris **241** (1955), 1667–1673.

Δ: ensemble abstrait. B_1 et B_2: familles de sous-ensembles de Δ, closes relativement à l'union et l'intersection finies, et telles que

$$((h_1 \in B_1)\&(h_2 \in B_2)) \to ((h_1 - h_1 \cdot h_2 \in B_1)\&(h_2 - h_2 \cdot h_1 \in B_2)).$$

φ: fonction numérique définie sur $B = B_1 \cup B_2$, finie, non négative, non décroissante telle que 1) si h et h' appartiennent tous deux à B_1 ou tous deux à B_2, $\varphi(h \cup h') \leqq \varphi(h) + \varphi(h')$, l'égalité étant vérifiée si h et h' sont disjoints, 2) si h et h' appartiennent l'un à B_1 et l'autre à B_2, $(h \supset h') \to (\varphi(h-h') = \varphi(h) - \varphi(h'))$. Un ensemble $E \subset \Delta$ est dit „mesurable selon Eudoxe" si à tout $\varepsilon > 0$ correspondent $h_1 \in B_1$ et $h_2 \in B_2$ tels que $h_1 \subset E \subset h_2$, $\varphi(h_2) - \varphi(h_1) < \varepsilon$. Si cette condition est satisfaite, la valeur commune de $\sup \varphi(h_1)$ pour $h_1 \in B_1$, $h_1 \subset E$ et de $\inf \varphi(h_2)$ pour $h_2 \in B_2$, $h_2 \supset E$ est désignée par $\varphi_J(E)$. Il est montré que φ_J est une mesure jordanienne (entendant „additive au sens fini"). Si $B_1 = B_2 = B$ et si φ, qui est alors une mesure jordanienne, est supposée dénombrablement additive, alors φ_J est une mesure dénombrablement additive prolongeant $\varphi|B$. *C. Pauc* (Nantes).

Šun, M. S. On the theory of singular integrals. Har'kov. Gos. Univ. Uč. Zap. **40** = Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) **23** (1952), 143–145. (Russian)

The author gives necessary and sufficient conditions for

$$\int_a^b f(t) d_t \Phi_n(x, t)$$

to converge, as $n \to \infty$, to $\alpha f(x+0) + \beta f(x-0)$ et each point x where f has a jump; α and β denote here positive numbers of sum 1. *A. Zygmund* (Chicago, Ill.).

See also: Milkman, p. 1114; Ohmann, p. 1124.

Functions of a Complex Variable, Generalizations

Griffith, James L. Hankel transforms of functions zero outside a finite interval. J. Proc. Roy. Soc. New South Wales **89** (1955), 109–115 (1956).

Necessary and sufficient conditions are given that a function be the Hankel transform of another function which is zero outside a finite interval.

Let $x^{\frac{1}{2}} g(x) \in L^2(0, \infty)$ and let the Hankel transform of g be

$$G(u) = \int_0^\infty x J_\nu(ux) g(x) dx,$$

where the integral is taken in the mean square sense.

Then $g(x) = 0$ $(x > A)$ if and only if (i) $G(z)$ is analytic in $0 \leqq \arg z \leqq \pi$, $|z| > \varepsilon > 0$, (ii) $z^{\frac{1}{2}} G(z) = O(e^{A \sin z})$ as $|z| \to \infty$, $\operatorname{Im} z \geqq 0$, (iii) $G(ue^{i\pi}) = e^{i\nu\pi} G(u)$, $u > 0$, (iv) $u^{\frac{1}{2}} G(u) \in L^2(0, \infty)$, (v) $|G(z)| = O(|z|^\nu)$ as $z \to 0$. *F. Goodspeed.*

Constantinescu, Corneliu. Quelques applications du principe de la métrique hyperbolique. Acad. R. P. Romîne. Stud. Cerc. Mat. **6** (1955), 529–566. (Romanian. Russian and French summaries)

„Cette note indique plusieurs applications du principe de la métrique hyperbolique. L'Auteur y démontre le grand théorème de Picard, le théorème de Julia et en indique un raffinement. Il y généralise le théorème de déformation de Koebe et améliore le théorème de Landau$\cdots$" (From the author's summary.) *M. H. Heins.*

Nevanlinna, Rolf. Sur la déformation dans la théorie de la représentation conforme. J. Math. Pures Appl. (9) **35** (1956), 109–114.

An extension of Ahlfors' distortion theorem is given. The starting point is a continuously differentiable univalent map $z = z(\sigma, \tau)$ with positive Jacobian having as its domain $[0 \leqq \sigma \leqq \sigma_0, 0 \leqq \tau \leqq \tau_0]$. Let G denote the image. Let $w(z)$ map the interior of G conformally and univalently onto the rectangle $[0 < u < a, 0 < v < b]$ in such a manner that $w[z(\sigma, \tau)]$ carries the vertices $(0, 0)$, $(\sigma_0, 0)$, $(0, \tau_0)$, (σ_0, τ_0) into $(0, 0)$, $(a, 0)$, $(0, b)$, (a, b) respectively. The quantity

$$\delta(\sigma) = \int_0^{\tau_0} \left| \frac{\partial z}{\partial \tau} \right|^2 \frac{d\tau}{\Delta}$$

is introduced. Here Δ is the Jacobian of $z(\sigma, \tau)$. It is shown that

$$b^2 \int_0^{\sigma_0} \frac{d\sigma}{\delta(\sigma)} \leqq \int_0^{\sigma_0} \frac{b^2 + \omega^2}{\delta(\sigma)} d\sigma \leqq ab,$$

where $\omega(\sigma) = \operatorname{osc}_{0 \leqq \tau \leqq \tau_0} \Re[w(z(\sigma, \tau))]$. Related results are given. [Cf. Nevanlinna, Lectures on functions of a complex variable, Univ. of Michigan Press, 1955, pp. 65–70; MR **16**, 1097.] *M. H. Heins* (Princeton, N.J.).

Reich, Edgar. On a Bloch-Landau constant. Proc. Amer. Math. Soc. **7** (1956), 75–76.

Landau proved [Math. Z. **30** (1929), 608–634] that the map of $|z| < 1$ by any function $f(z) = z + a_2 z^2 + \cdots$, schlicht in $|z| < 1$, contains an open disk of radius .566.

This numerical value is now improved to .569. No details of the computation involved are given.

W. W. Rogosinski (Newcastle-upon-Tyne).

★ **Wittich, Hans. Neuere Untersuchungen über eindeutige analytische Funktionen.** Ergebnisse der Mathematik und ihrer Grenzgebiete (N.F.), Heft 8. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. iv+163 pp. DM 25.60.

The present monograph is an account of researches of the past two decades concerning meromorphic functions with principal emphasis on the Nevanlinna theory and its applications. Reference to some of this large corpus of work is to be found in the second edition of Nevanlinna's "Eindeutige analytische Funktionen" [Springer, Berlin, 1953; MR **15**, 208], however lack of space prohibited an account of much of this work in the Nevanlinna monograph. For this reason, the present volume of Wittich constitutes a valuable addition to the monographic literature on the theory of meromorphic functions and one which should be of great utility to workers in this field. In conformity with its objective, proofs are given in greater detail than in the case in a summarizing report. Attention is concentrated on meromorphic functions in plane domains.

Central consideration is accorded to the recent developments in the following areas of investigation: the reversal of the second fundamental inequality (Teichmüller, H. Selberg, Collingwood); the application of the Nevanlinna theory to ordinary differential equations; the type problem; the realization problem of "Wertverteilungslehre". The exposition is enriched by a large collection of illustrative examples.

The first chapter treats the Wiman-Valiron theory of the maximal term of a power series which is an important tool in the analytic theory of ordinary differential equations. It culminates in a proof of the small Picard theorem. Chapter II gives an account of the two fundamental theorems of the Nevanlinna theory. Chapter III is an exposition of various supplementary material pertaining to the fundamental theorems. Among the topics treated are: relations between growth and the distribution of defective values; the examples of Teichmüller, Hayman and Mme. Laurent Schwartz concerning defective non-asymptotic values, Lehto's "Prinzip der Anzahlfunktionen"; the extension of the Nevanlinna theory to multiply-connected regions (G. af Hällström). Chapter IV investigates sufficient conditions on a meromorphic function for the second fundamental theorem to take the form of an asymptotic equality outside of the usual exceptional sets (Umkehrung des zweiten Hauptsatzes). One condition (in the case where the domain is the finite plane) is that the function satisfy a differential equation of the form $p_m w^{(m)}+\cdots+p_0 w=0$ ($p_0\not\equiv 0$), where the p_k are polynomials. A second sufficient condition is that the Riemannian image be ramified over only a finite number of points and satisfy elsewhere certain obvious conditions (F_q surfaces). Extensions of this result due to H. Selberg and Collingwood are discussed. Chapter V brings the machinery of the Nevanlinna theory to bear on the study of the solutions of ordinary differential equations. A considerable part of this work is due to the author. Among the topics treated are: solutions of non-linear differential equations which are entire functions, order and defect for solutions of linear differential equations, the theorem of Malmquist, the Riccati and Painlevé differential equations. Chapter VI, which treats the conformal and quasi-conformal mapping of annuli, has as its object the introduction of methods for studying F_q surfaces. The type problem for Riemann surfaces is studied in Chapter VII for F_q surfaces .The Nevanlinna-Wittich criterion is treated. Special classes of surfaces are considered. An account of the status of the realization problem (Umkehrproblem der Wertverteilung) is given in Chapter VIII. This problem is: given a sequence of points $\{a_k\}$ of the extended plane and sequences $\{\delta_k\}$, $\{\theta_k\}$ of non-negative numbers satisfying $\delta_k+\theta_k\leqq 1$ for each k and $\sum\delta_k+\sum\theta_k=2$ to determine a meromorphic function w in the finite plane satisfying $\delta(w, a_k)=\delta_k$, $\theta(w, a_k)=\theta_k$. A discussion is given of the work in this direction of Nevanlinna, E. Ullrich, Teichmüller, Wittich, Huckemann and Künzi. The final chapter treats analytic functions with bounded Dirichlet integral. Here interpolation problems (Lokki), slit maps (Schiffer), D-removable sets (Lehto, Ahlfors-Beurling) and the connection with extremal length are considered.

M. Heins (Princeton, N.J.).

Radojčić, M. Entwicklung analytischer Funktionen auf Riemannschen Flächen nach algebraischen oder gewissen endlich vieldeutigen transzendenten Funktionen. Acad. Serbe Sci. Publ. Inst. Math. 8 (1955), 93–122.

The author is concerned with generalizations of the Runge, Mittag-Leffler and Weierstrass theorems to Riemann surfaces. A Riemann surface R is said to be contained in a limiting sense (grenzweise enthalten) in a sequence of compact Riemann surfaces $\{A_n\}$ provided that each relatively compact region of R is a region of A_n for n sufficiently large. An approximation theorem for functions analytic on a region G of a Riemann surface R is established where the approximants are meromorphic functions on the A_n of a sequence containing a region G' ($\supseteq G$) in a limiting sense. The surfaces R and the A_n are taken as Riemannian coverings of the extended plane. Existence theorems of the Mittag-Leffler and Weierstrass types are also established. Other related results are also given.

M. H. Heins (Princeton, N.J.).

★ **Cartwright, M. L. Integral functions.** Cambridge tracts in mathematics and mathematical physics, No. 44. Cambridge, at the University Press, 1956. viii+135 pp. $3.50.

This useful book deals primarily with the theory of analytic functions (whether entire or not) which are of finite order in an angle. Only the principal results are given, but even these have never before been collected in one place. Little account is taken of developments since about 1940, but most of the theory dealt with here was already in final form by that time. The first two chapters are introductory, dealing with such matters as various versions of the Poisson-Jensen formula and Carleman's formula, and with generalities about entire functions of finite order. The third chapter is a detailed account of Phragmén-Lindelöf theorems and the Phragmén-Lindelöf function $h(\theta)$. Chapter 4 gives a detailed account of Lindelöf proximate orders and shows in particular how they make the classical minimum modulus theorems almost obvious. The next two chapters deal in detail, and in the full generality that comes from using proximate orders, with the (far from obvious) relationships between the growth of a function and the distribution of its zeros. Chapter 7 gives the main theorems on Julia lines. The final chapter discusses entire functions of exponential type, the indicator diagram and the Borel polygon, and culminates in theorems connecting the singular points of

the Borel-Laplace transform of an entire function with the exceptional values of the function itself.
R. P. Boas, Jr. (Evanston, Ill.).

Sikkema, P. C. A generalization of Nörlund's theory of principal solutions of linear difference equations. I, II. Nederl. Akad. Wetensch. Proc. Ser. A. **58** (1955), 608–620; **59** (1956), 83–94=Indag. Math. **17** (1955), 608–620; **18** (1956), 83–94.

The author continues his study of differential operators of infinite order, having constant coefficients. Let $F(D)=\sum a_nD^n$, $a_0\neq 0$, and construct $\phi(D)=\sum b_nD^n$ so that $F(D)\phi(D)=1$. Let $h(x)$ be an entire function of growth at most order 1, type 0, and consider the nonhomogeneous differential equation $F(D)y(x)=h(x)$. The author's main result is the following: Under suitable restrictions on $F(D)$, a solution of this equation is provided by $y(x)=\phi(D)h(x)$; this solution is of the same rate of growth as $h(x)$; and there is no solution of slower growth. The nature of $F(D)$ is restricted by the rate of growth of h. If $h(x)$ is of order $\sigma<1$, type τ, $0<\tau<\infty$, then $f(Z)=\sum (n!)^{-1/\sigma}a_nZ^n$ is required to be of exponential type at most $(\sigma\tau)^{-1/\sigma}$. When $\tau=0$, $f(Z)$ is required merely to be of order 1, finite type (and this holds also in this case for $\sigma=1$). If $0<\sigma<1$ and $\tau=\infty$, then f is required to be of order smaller than 1. Finally, when $h(x)$ is of zero order (and is not a polynomial) then it is required that $\sum (n!)^{-\nu}a_nZ^n$ be of finite order for sufficiently large ν. [Reference should also be made to Gel'fond, Trudy Mat. Inst. Steklov. **38** (1951), 42–67, and to the additional references there; MR **13**, 929; **14**, 739.] *R. C. Buck.*

Sikkema, P. C. Conditions for applicability of linear differential operators of infinite order with polynomial coefficients. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18** (1956), 181–189.

The author extends some results of his thesis [Differential operators and differential equations of infinite order ···, Groningen, 1953; MR **15**, 623] to differential operators with polynomial coefficients,

$$F(x, D)=\sum_{n=0}^{\infty}(a_{0n}+a_{1n}x+\cdots+a_{kn}x^k)D_n.$$

A typical result is that $F(x, D)$ is applicable to all entire functions of positive order σ and normal type τ, if and only if

$$\limsup\,\{n!^{1-1/\sigma}|a_{\mu n}|\}^{1/n}\leqq(\sigma\tau)^{-1/\sigma}\ (\mu=0, \cdots, k).$$

R. P. Boas, Jr. (Evanston, Ill.).

Clunie, J. The derivative of a meromorphic function. Proc. Amer. Math. Soc. **7** (1956), 227–229.

It is well-known that the derivative of a meromorphic function of finite order is of the same order as the function itself [G. Valiron, Lectures on the general theory of integral functions, Privat, Toulouse, 1923; also J. M. Whittaker, J. London Math. Soc. **2** (1936), 82–87]. An equivalent result, as pointed out by Whittaker, states that if $f(z)$ and $g(z)$ are two integral functions of orders ϱ_1 and ϱ_2 with $\varrho_1>\varrho_2$, then $f'(z)g(z)-f(z)g'(z)$ is of order ϱ_1. The author gives a new proof for this theorem which, unlike the previous proofs, is independent of meromorphic function theory, but which depends entirely on integral function theory. *M. S. Robertson.*

Hiong, King-Lai. Nouvelle démonstration et amélioration d'une inégalité de M. Milloux. Bull. Sci. Math. (2) **79** (1955), 135–160.

The author carries out the emendation suggested in the review of a previous paper [same Bull. (2) **78** (1954), 181–198; MR **16**, 460]. The effort seems somewhat disproportionate to the result. *W. K. Hayman* (London).

Tumarkin, G. C. On uniform convergence of certain sequences of functions. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 1151–1154. (Russian)

The following theorem is established: Let $\{f_n\}$ denote a sequence of functions analytic in $|z|<1$ and satisfying

$$\int_0^{2\pi}\log^+|f_n(re^{i\theta})|d\theta\leqq C\quad(0<r<1;\ n=1, 2, \cdots).$$

Suppose that $\{f_n(e^{i\theta})\}$ converges in a set E of positive measure. Then there exists a subsequence $\{f_{n(i)}\}$ having the property that for any $\varepsilon>0$ there exists a perfect subset P of E, $mP>mE-\varepsilon$, such that $\{f_{n(i)}\}$ converges uniformly on each set $D(P_\gamma)$, where $D(P_\gamma)$ is defined for $0<\gamma\leqq\pi/2$ as the set obtained from $|z|<1$ by removing the open lunes bounded by the arcs of $|z|=1$ contiguous to P and the circular arcs with the same endpoints which make an angle γ with $|z|=1$. [For an anterior result in this direction, cf. Zygmund, Fund. Math. **36** (1949), 207–235, p. 213; MR **12**, 18.] *M. Heins* (Princeton, N.J.).

Tumarkin, G. C. On sequences of meromorphic functions with uniformly bounded areas of Riemann surfaces over a sphere. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 199–202. (Russian)

This paper establishes a number of theorems concerning sequences (or families) of meromorphic functions with Riemannian images having uniformly bounded spherical areas. Let $A(f)$ denote the spherical area of the Riemannian image of a function f meromorphic in $|z|<1$. It is shown that if a sequence of functions $\{f_n\}_1^\infty$, each meromorphic in $|z|<1$, satisfies $\sup_n A(f_n)<+\infty$ and converges uniformly in $|z|<1$ less a finite number of irregular points, then

$$\lim_{n\to\infty}\int_0^{2\pi}[f_n(e^{i\theta}), f(e^{i\theta})]d\theta=0,$$

where f is the limit function of the sequence. Here $[a, b]$ denotes the spherical distance between a and b. Further $f_n(e^{i\theta})$ converges in measure to $f(e^{i\theta})$. Application is made of these results to compactness questions pertaining to the family of boundary functions of the meromorphic functions f satisfying $A(f)\leqq C$. The arguments used in the paper depend upon the notion of quasinormal families and simple area-length estimates as well as the Lusin-Privaloff uniqueness theorem. *M. H. Heins* (Princeton, N.J.).

Umezawa, Toshio. On the theory of univalent functions. Tôhoku Math. J. (2) **7** (1955), 212–228.

The author obtains a number of theorems involving sufficient conditions that a function $f(z)$ regular or meromorphic in a given region be univalent or multivalent in the region. The notion of convexity or partial convexity of the image curve is used extensively. A few of the theorems obtained are mentioned below.

The theorem of K. Noshiro [Proc. Imp. Acad. Tokyo **8** (1932), 275–277] and J. Wolff [C. R. Acad. Sci. Paris **198** (1934), 1209–1210] is generalized as follows. Let $f(z)$ be regular and $f'(z)\neq 0$ in a closed convex domain D with a regular curve L as boundary. Let z_i be the roots of $d\arg f(z)/d\arg dz=0$ on L. If $\Re\, e^{i\alpha}f'(z_i)>0$ for all z_i (α a real constant), then $f(z)$ is univalent in D.

Let $f(z)=z+a_2z^2+\cdots$ be regular for $|z|\leqq 1$ and $|f''(z)|<6^{\frac{1}{2}}|f'(z)|$ in $|z|\leqq 1$. Then $f(z)$ is univalent in $|z|\leqq 1$.

Let $f(z)=z^{-1}+a_0+a_1z+\cdots$ be regular for $0<|z|\leqq 1$, and let $-2<1+\Re[zf''(z)/f'(z)]<0$ on $|z|=1$. Then $f(z)$ is univalent in $|z|\leqq 1$.

Let $f(z)$ be regular and single-valued in a closed convex domain D which has $n-1$ circular holes $|z-a_i|<r_i$ in it. Suppose that for each i

$$0<1+\Re(z-a_i)\frac{f''(z)}{f'(z)}<2 \text{ on } |z-a_i|=r_i,$$

and $\Re e^{i\alpha}f'(z)>0$ in D (α a real constant). Then $f(z)$ is univalent in D.

Let $f(z)$ be regular in a closed convex domain D whose boundary L is a regular curve. Suppose that $f(z)$ has exactly $p-1$ critical points α_i in D and no critical point on L. If $\Re[e^{i\alpha}f'(z)/\prod(z-\alpha_i)]>0$ on L, then $f(z)$ is at most p-valent in D.

Let $f(z)=z^p+a_{p+1}z^{p+1}+\cdots$ be regular for $|z|\leqq 1$ and let $|f(z)/pz^{p-1}|<M$ for $|z|\leqq 1$. Then $f(z)$ is p-valent in $|z|<1/M$ and $f(z)$ maps $|z|<M_2-(M_2^2-1)^{\frac{1}{2}}$ onto a p-valently convex domain where

$$M_2=\tfrac{1}{2}\Big[M\Big(1+\frac{1}{p}\Big)+\frac{1}{M}\Big(1-\frac{1}{p}\Big)\Big].$$

M. S. Robertson (New Brunswick, N.J.).

Kufarev, P. P. On a certain method of investigation of extremum problems in the theory of univalent functions. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 633–635. (Russian)

Let $\psi(w)=w+\cdots$ be regular and univalent in $|w|<1$ and let w_0 be a fixed point in $|w|<1$, and θ a fixed real number. The author determines implicitly the extreme values for $\Re[e^{i\theta}\log\psi'(w_0)]$. He also announces sharp bounds for $\Re[e^{i\theta}\psi(w_0)]$ for this class, and sharp bounds for $\Re[e^{i\theta}\psi'(w_0)]$ for $\psi(w)=w+\cdots$ regular and univalent in $|w|>1$.

The author seems to be unaware of the work of Grad contained in the last chapter of the book by Schaeffer and Spencer [Coefficient regions for schlicht functions, Amer. Math. Soc. Colloq. Publ., v. **35**, New York, 1950; MR **12**, 326]. *A. W. Goodman* (Princeton, N.J.).

Danilevskiĭ, A. M. On single-valued univalent functions on an annulus. Har'kov. Gos. Univ. Uč. Zap. **34**= Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) **22** (1950), 51–63 (1951). (Russian)

Let R_q denote the class of functions $f(z)$ regular and univalent in $q\leqq|z|<1$ $(0\leqq q<1)$ and mapping $|z|=q$ into $|w|=q$, preserving direction. If A is the area of the image domain, the author proves that $A\geqq\pi(1-q^2)$ with equality if and only if $f(z)=e^{i\alpha}z$. He also obtains for the class R_q, a Lindelof principle, a Koebe theorem for the outer boundary of the image domain, and a sharp lower bound for $|f(z_0)|$. In the last two results, the extremal function is an elliptic function which maps the ring onto the region $|w|\geqq q$ except for a radial slit extending to infinity.

A. W. Goodman (Princeton, N.J.).

Marčenko, A. R. Some extremal problems in the theory of univalent functions. Leningrad. Gos. Univ. Uč. Zap. **144.** Ser. Mat. Nauk **23** (1952), 257–269. (Russian)

Let $\varphi(z,h,\theta)$ map $|\zeta|<1$ conformally onto the unit circle with a radial slit of length ε issuing from $e^{i\theta}$, preserving directions at the origin, where $h=\varepsilon^2/(2-\varepsilon)^2$. The author examines various compositions of φ and functions $f(z)=z+\sum_{n=2}^{\infty}a_nz^n\in S$, the class of functions regular and univalent in $|z|<1$. He proves (I) if $f(z)=z/(1-z)^2$ and $f_{(1)}(z)=f(\varphi(z,h,\theta))/c=z+\sum_{n=2}^{\infty}a_n^{(1)}z^n$, then for sufficiently small ε, $|a_n|\leqq n$ with equality only at $\theta=\pi$. (II) If $f(z)\in S$ and $|a_n|$ is maximal for this class, then $(n-1)r_n=-\sum_{j=1}^{n-1}jr_j\cos(\varphi_j-\varphi_n)$ and $\sum_{j=1}^{n-1}j(n-j)\sin(\varphi_j-\varphi_n)=0$, where $a_j=r_je^{i\varphi_j}$.

The paper contains quite a few misprints and obscure points. *A. W. Goodman* (Princeton, N.J.).

Reade, Maxwell. On the coefficients of certain univalent functions. Ann. Acad. Sci. Fenn. Ser. A. I. no. **215** (1956), 6 pp.

Generalizing a result of A. Rényi [Publ. Math. Debrecen **1** (1949), 18–23; MR **11**, 92] the author obtains the following theorem. Let $f(z)=z+\sum_2^{\infty}a_nz^n$ be analytic and $f'(z)\neq 0$ in $|z|<1$. If

$$\text{(1)}\qquad \int_{\theta_1}^{\theta_2}\Re\left\{1+re^{i\theta}\frac{f''(re^{i\theta})}{f'(re^{i\theta})}\right\}d\theta>-\frac{\pi}{2}$$

holds for all $\theta_1<\theta_2$ and for all $0\leqq r<1$, then $f(z)$ is univalent and the coefficients satisfy (2) $|a_n|\leqq\frac{1}{2}(n+1)$ $(n=1,2,\cdots)$. (2) is shown to be sharp for $n=2$. From a result of W. Kaplan [Michigan Math. J. **1** (1952), 169–185; MR **14**, 966] (1) implies the existence of a univalent convex function $\phi(z)$ such that

$$\left|\arg\frac{f'(z)}{\phi'(z)}\right|\leqq\frac{\pi}{4}\ (|z|<1).$$

An analogous result is obtained for a sub-class of the class of near-to-star functions introduced in an earlier paper by the author [M. Reade, ibid. **3** (1955), 59–62; MR **17**, 25]. This result sharpens inequalities for the coefficients of a sub-class of the reviewer's functions starlike in one direction [M.S. Robertson, Amer. J. Math. **58** (1936), 465–472]. *M. S. Robertson.*

Royster, W. C. Rational univalent functions. Amer. Math. Monthly **63** (1956), 326–328.

Using a method of Linis [same Monthly **62** (1955), 109–110; MR **16**, 809] the author enumerates those functions regular and univalent in the unit circle with expansion at the origin $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$ for which all coefficients a_n are in a quadratic field $R(m^{\frac{1}{2}})$ where m is 1 or a square-free negative integer thus obtaining results due to Friedman [Duke Math. J. **13** (1946), 171–177; MR **8**, 22] and Bernardi [ibid. **19** (1952), 5–21; MR **13**, 733]. Lemma 2 can be proved more easily by observing that $f(z)$ is regular if and only if the meromorphic univalent function $(f(z^{-1}))^{-1}$ is non-zero. *J. A. Jenkins.*

Dundučenko, L. O. On some properties of analytic functions belonging to special classes. Dopovidi Akad. Nauk Ukrain. RSR **1956**, 119–123. (Ukrainian. Russian summary)

Let $w(z)=z^p+\sum_{n=2}^{\infty}b_nz^{n+p}$ be regular and p-valent in $|z|<1$, and map this circle onto a domain that is convex in the generalized sense. The author announces sharp bounds for $|b_n|$, $|w(re^{i\theta})|$, $|w'(re^{i\theta})|$, $|\arg w'(re^{i\theta})|$, and for the radius of curvature of the image curve of $|z|=r$. Similar theorems are stated for the case of a starlike image domain, and extensions are indicated to functions with s-fold symmetry.

Let $f(z)=z+\sum_{n=2}^{\infty}c_nz^n$ be regular, univalent, and $|f(z)|<R$ in $|z|<1$. The author announces that

$$(n-1)|c_n|<P_{n-1}(\cos\alpha)+P_{n-2}(\cos\alpha)+\cdots+P_0(\cos\alpha),$$

where P_n is the nth Legendre polynomial, and $\cos\alpha=(R-2)/R$.

Several of the theorems on p-valent functions were proved earlier by the reviewer [Trans. Amer. Math. Soc. 68 (1950), 204–223; MR 11, 508]. *A. W. Goodman.*

Fišman, K. M. On a class of Hilbert spaces of analytic functions. Dokl. Akad. Nauk SSSR (N.S.) 107 (1956), 24–27. (Russian)

The author considers the Hilbert space $Z_E{}^2$ of analytic functions $\sum a_nz^n$ for which $\sum |a_n|^2\alpha_n<\infty$, where α_n are positive and finite and $E(z)=\sum z_n/\alpha_n$ has the unit circle as its circle of convergence; this is a generalization of H_2. If α_n are the $(2n)$th moments over $(0, 1)$ of an integrable function that is bounded from 0 and ∞ on every subinterval of $(0, 1)$, we obtain $Z_{E_\sigma}{}^2$, which is a more direct generalization of H_2. A necessary and sufficient condition for $f\in Z_{E_\sigma}{}^2$ is that $\int_0^1\int_0^{2\pi}\sigma(\varrho)|f(\varrho e^{i\theta})|^2d\varrho d\theta<\infty$. The author discusses transformations from one $Z_E{}^2$ to another, and integral representations. He gives formulas for calculating $f(z)$ from the values $f^{(n)}(\beta_n)$, using the functions

$$g_n(z,\beta_n)=z^nE^{(n)}(\bar\beta_nz).$$

R. P. Boas, Jr. (Evanston, Ill.).

Fišman, K. M. On completeness of some systems of analytic functions. Dokl. Akad. Nauk SSSR (N.S.) 107 (1956), 205–208. (Russian)

The author considers the Hilbert space $Z_E{}^2$ of the preceding review, but with $E(z)$ now an entire function. Using Hilbert space methods and special properties of the space, he establishes the following results. Let $g(z)=\sum a_nz^n$, $a_n\neq 0$. (1) If the numbers $e^{i\theta_n}$ are distinct, then the sequence $g_k=\sum a_ne^{ik\theta_n}z^n$ $(k=0, \pm1, \pm2, \cdots)$ is complete in the set of all entire functions. (2) If $g_k(z)=\sum a_nb_n{}^kz^n$ $(k=0, 1, 2, \cdots)$, the same conclusion holds if the b_n are either real or have at most a finite number of limit points, which are not members of $\{b_n\}$. This is a generalization of a theorem of Ganapathy Iyer [J. Indian Math. Soc. (N.S.) 17 (1953), 183–185; MR 15, 719]. (3) The same conclusion holds if b_n are distinct nonnegative numbers, and $g_k(z)=\sum a_nb_n{}^ke^{-b_n}z^n$; or if b_n are distinct real numbers and $g_k(z)=\sum a_nb_n{}^ke^{-b_n{}^2}z^n$. Finally the author indicates extensions to functions analytic in the unit disk and to the case of repeated elements in $\{b_n\}$.

R. P. Boas, Jr. (Evanston, Ill.).

Kolbina, L. I. On distortion theorems for certain classes of p-valent functions. Vestnik Leningrad. Univ. 11 (1956), no. 7, 71–76. (Russian)

Let $f(\zeta)=\zeta^p(1+a_1\zeta+\cdots)$ be regular and p-valent in $|\zeta|<1$, and suppose that $2\pi L\equiv\int_{|\zeta|=1}\log|f(\zeta)|d\arg f(z)$ is finite. Then for a fixed z, $|z|<1$, $f(z)\leq e^{L/2p}|z|^p/(1-|z|^2)^p$ with equality only for the function $f^*(\zeta)=\zeta^p/(1-\bar z\zeta)^{2p}$. The author extends this theorem in a variety of ways.

A. W. Goodman (Princeton, N.J.).

Goluzin, G. M. On a variational method in the theory of analytic functions. Leningrad. Gos. Univ. Uč. Zap. 144. Ser. Mat. Nauk 23 (1952), 85–101. (Russian)

The author first derives two general variational formulas for any class of functions with a representation $f(z)=\int_a^b g(z, t)d\alpha(t)$, where $g(z, t)$, $a<b$ are fixed and $\alpha(t)$ is a real-valued non-decreasing function in $a\leq t\leq b$ with the usual normalization. He then makes a large variety of applications to some particular classes, namely functions that are starlike in $|z|<1$, typically real functions, and functions mapping $|\zeta|\geq1$ onto the complement of a starlike region. Here is a typical one of his many theorems.

Let $\Phi(w)$ be an entire function and let S^* be the class of functions $f(z)=z+\cdots$ regular, univalent and starlike in $|z|<1$. Then $\max\Re(\Phi(\log f(z)/z))$ and $\max|\Phi(\log f(z)/z)|$ for $f(z)\in S^*$ are attained for a function of the form $z/(1-e^{i\alpha}z)^2$ where α is real and depends on z and Φ. Here the case in which $\Phi'(\log f(z)/z)=0$ for the extremal function is excluded from consideration.

A. W. Goodman (Princeton, N.J.).

Cowling, V. F. On analytic functions having a positive real part in the unit circle. Amer. Math. Monthly 63 (1956), 329–330.

The author rediscovers the proof of Borel's Theorem on the coefficients of functions with positive real part, which is given in Littlewood's "Lectures on the theory of functions" [Oxford, 1944, p. 114;MR 6, 261]. *W. K. Hayman.*

Ricci, Giovanni. Su un problema di massimo per le funzioni maggioranti delle serie di potenze. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 18 (1955), 609–613.

Let $f(z)=\sum_{n=h}^\infty a_nz^n$, where $\sum_{n=h}^\infty|a_n|^2\leq1$. The author proves that if r_h is the positive root of the equation $r^{2h}+r^2-1=0$, then $\mathfrak{M}(r_h)=\sum|a_n|r_h{}^n\leq1$. The examples

$$f_h(z)=z^h\surd(1-r_h{}^2)/(1-r_hz)$$

show that the constants r_h can not be increased in this theorem. *G. Piranian* (Ann Arbor, Mich.).

Tanaka, Chuji. Note on Dirichlet series. XIV. On the singularities of Dirichlet series. VI. Tôhoku Math. J. (2) 7 (1955), 229–239.

Let $F(s)=\sum_{n=1}^\infty a_n\exp(-\lambda_ns)$ $(s=\sigma+it)$, $0\leq\lambda_1<\lambda_2<\cdots<\lambda_n\to+\infty$. We assume throughout that this series is simply convergent in the half plane $\sigma>0$. The point s_0 on the abscissa of convergence $\mathrm{Re}\,s_0=0$ is said to be a Picard point if $F(s)$ assumes every value except perhaps two (∞ included) infinitely many times in every half circle $|s-s_0|<\varepsilon$, $\sigma>0$. In this paper the author gives various sufficient conditions for a point s_0 to be a Picard point. For example, if $|\arg a_n|\leq\theta<\pi/2$, and if there exists a sequence $\{x_k\}$ $(0<x_k\uparrow\infty)$ such that

$$\limsup_{k\to\infty} 1/x_k\cdot\log|\Delta_k{}^*|=0,$$

$$\limsup_{k\to\infty} 1/\log x_k\cdot\log^+\log^+|\Delta_k{}^*|>\tfrac{1}{2},$$

where

$$\Delta_k{}^*=\sum_{[x_k]\leq\lambda_n<x_k} a_n,$$

then $s=0$ is a Picard point. If $\lim\inf_{n\to\infty}(\lambda_{n+1}-\lambda_n)>0$ and if $\lim\sup 1/\log\lambda_n\cdot\log^+\log^+|a_n|=\frac{1}{2}+\alpha$ $(\alpha>0)$, and $\lim\sup_{n\to\infty}\log n/\log\lambda_n<\alpha$, then every point on $\sigma=0$ is a Picard point. *I. I. Hirschman* (Levallois).

Bremermann, H. J. On the conjecture of the equivalence of the plurisubharmonic functions and the Hartogs functions. Math. Ann. 131 (1956), 76–86.

We assume familiarity with the notions of a Hartogs function [Bochner and Martin, Several complex variables, Princeton, 1948, p. 143; MR 10, 366] and of a plurisubharmonic (also called pseudo-convex) function [Oka, Tôhoku Math. J. 49 (1942), 15–52; MR 7, 290]. Every upper semi-continuous Hartogs function is a plurisubharmonic function. The author proves that in a domain of holomorphy, it is true conversely that every pluri-

subharmonic function is a Hartogs function (and of course is upper semi-continuous). He also shows, by a counterexample, that this converse is not necessarily true without the condition that the domain be a domain of holomorphy; as a consequence, this disproves the conjecture of Bochner and Martin that a real C^2 function $f(z)$ for which the Hermitian form $\sum (\partial^2 f(z)/\partial z_\mu \partial \bar{z}_\nu)\, dz_\mu d\bar{z}_\nu$ is positive semi-definite must be a Hartogs function.

R. C. Gunning (Princeton, N.J.).

Roșculeț, M. N. Algèbres infinies associées à des équations aux dérivées partielles, homogènes, aux coefficients constants d'ordre quelconque. Acad. R. P. Romîne. Stud. Cerc. Mat. **6** (1955), 567–643. (Romanian. Russian and French summaries)

For every homogeneous partial differential equation of the form

$$\sum C_{\alpha_0,\alpha_1,\cdots,\alpha_p} \frac{\partial^n U}{\partial x_0^{\alpha_0} \partial x_1^{\alpha_1} \cdots \partial x_p^{\alpha_p}}$$

with constant coefficients the author determines a commutative, associative algebra of infinite order. The multiplication table of each such algebra as determined by the coefficients of the differential equation. [See P. W. Ketchum, Amer. J. Math. **54** (1932), 253–264.]

A study of the structure of such algebras is made. Monogenity is defined and studied. Solutions of the differential equation are determined by monogenic functions in the associated algebra.

Considerable attention is given to the case of the Laplacian differential equation in three independent variables.

J. A. Ward (Holloman, N.M.).

Roșculeț, Marcel. Algèbres infinies attachées à une classe d'équations aux dérivées partielles. Com. Acad. R. P. Romîne **5** (1955), 1245–1252. (Romanian. Russian and French summaries)

In order to solve the generalized Laplacian equation

$$\frac{\partial^2 U}{\partial x_0^2}+\frac{\partial^2 U}{\partial x_1^2}+\cdots+\frac{\partial^2 U}{\partial x_n^2}+\frac{\partial^2 U}{\partial y^2}=0$$

the author develops a commutative, associative algebra A of infinite order. Solutions are obtained from functions monogenic in A. The procedure is similar to that of P. W. Ketchum [Amer. J. Math. **51** (1929), 179–188].

J. A. Ward (Holloman, N.M.).

See also: Boreli, p. 1148 (twice); Kravtchenko, Saint-Marc, Sauvage and Boreli, p. 1148; Kuramochi, p. 1072; Walsh, p. 1077.

Analytic Functions of Several Variables

Stoilow, S. Remarques sur le principe des extrema et sur ses applications en théorie des fonctions. Acad. R. P. Romîne. Fil Iași. Stud. Cerc. Ști. **6** (1955), 13–18. (Romanian. Russian and French summaries)

Etude d'une norme $|w|_\alpha$ attachée à tout nombre complexe et obtenue comme quotient du groupe (multiplicatif) des nombres complexes par un sous groupe g. On a en particulier $|w|_\alpha=|w|$ pour $\alpha=\frac{1}{2}\pi$ et $|w|_\alpha=\arg w$ pour $\alpha=\pi$. Si w est holomorphe en z, $|w|_\alpha$ est harmonique d'où l'application du principe du maximum et l'obtention d'inégalités analogues au lemme de Schwarz et portant (comme cas particulier) sur $\arg w(z)$ au lieu de $|w(z)|$.

L. Fourès (Marseille).

Morse, Marston. La construction topologique d'un réseau isotherme sur une surface ouverte. J. Math. Pures Appl. (9) **35** (1956), 67–75.

Soit S une surface ouverte, M son recouvrement universel, Γ le groupe des homeomorphismes g de M sur M tel que p et $g(p)$ recouvrent le même point de S. Soit f continue à valeurs complexes, définie sur S: les lignes de niveau de $\Re f$ et $\Im f$ constituent le réseau isotherme (F_f, G_f) de f. La présente note a pour objet de donner une caractérisation topologique des réseaux isothermes des fonctions f sur S. A une structure conforme K sur S correspondent les fonctions K-analytiques et un réseau (F, G) sur S sera K-isotherme si localement c'est le réseau isotherme d'une fonction K-analytique. Le résultat fondamental de la première partie indique que tout réseau K-isotherme est la projection sur S du réseau isotherme d'une certain fonction λ intérieure sur M vérifiant en outre $\lambda(g(p))=a_g\lambda(p)+b_g$. Un réseau topologique (F, G) est localement le réseau isotherme d'une fonction intérieure, et il est isothermiquement réalisable s'il existe une structure conforme K sur S telle que (F, G) soit K-isotherme. La deuxième partie est consacrée à la recherche de conditions globales sur F et G telles que (F, G) soit isothermiquement réalisable. Le cas de la sphère privée de deux points fournit un example particulièrement intéressant de recherche de ces conditions. Il est à noter que cette note n'est qu'une introduction à une théorie en cours de développement.

L. Fourès (Marseille).

Agrusti, Giovanni. Sui raggi associati di convergenza di una serie di potenze ad n variabili complesse. Giorn. Mat. Battaglini (5) 3(83) (1955), 223–296 (1956).

Call r_1 and r_2 associated radii of convergence of the power series $P(x_1, x_2)=\sum_{\mu,\nu} a_{\mu\nu}x_1^\mu x_2^\nu$ if the series $P(x_1, x_2)$ is absolutely convergent for $|x_1|<|r_1|$ and $|x_2|<r_2$ but is absolutely divergent for $|x_1|>r_1$ and $|x_2|>r_2$. Faber [Math. Ann. **61** (1905), 289–324] has shown that the set of points (r_1, r_2) constitutes a curve $\varphi(r_1, r_2)=0$ which has a forward and backward tangent at each point and is the solution curve of a second-order differential equation. Also corresponding to the solution curves of such equations are power series $P(x_1, x_2)$ for which such a curve is the curve associated with such radii of convergence. The present author extends much of the work of Faber to power series in three variables,

$$P(x_1, x_2, x_3)=\sum a_{\mu\nu\rho}\, x_1^\mu x_2^\nu x_3^\rho.$$

The results are too involved to be presented here.

V. F. Cowling (Lexington, Ky.).

Sommer, Friedrich; and Mehring, Johannes. Kernfunktion und Hüllenbildung in der Funktionentheorie mehrerer Veränderlichen. Math. Ann. **131** (1956), 1–16.

Si $\mathfrak{G}$ est un domaine de Reinhardt de centre l'origine dans C^2, pour que $f \in L^2(\mathfrak{G})$ appartienne aussi à la classe $L^2[\mathfrak{H}(G)]$ des fonctions holomorphes de carré sommable dans la cellule d'holomorphie $\mathfrak{H}(G)$, il faut et il suffit qu'on ait $a_{mn}^* \le k a_{mn}$ pour tous les couples, m, n d'entiers pour lesquels $a_{m,n}$ est fini:

$$a_{mn}^2=\int_{\mathfrak{G}} |z_1|^{2m}|z_2|^{2n}d\omega, \quad a_{mn}^{*2}=\int_{\mathfrak{H}(G)} |z_1|^{2m}|z_2|^{2n}d\omega.$$

Exemple d'un domaine de Reinhardt dans C^2 avec $L^2[\mathfrak{H}(G)]=L^2(G)$ et d'un domaine de même type $\mathfrak{G}$ avec $f(z) \in L^2(\mathfrak{G})$, et $|f|^2$ non sommable dans $\mathfrak{H}(G)$.

Si $A(G)$ est l'intersection des domaines d'holomorphie contenant G et sa frontière, le domaine $K(G)$ dans lequel le noyau $K_G(z)=K_G(z, \bar{z})$ est prolongeable comme fonction analytique réelle satisfait à $\mathfrak{H}(G)\subset K(G)\subset A(G)$, ceci dans C^n, n quelconque. Exemple dans C^2, avec $G=[0\leq|z_1|\leq|z_2|\leq 1]$, $A(G)-K(G)$ contenant un ouvert.

Si G est borné et univalent le noyau $K_G(z, \bar{\zeta})$ de S. Bergmann est holomorphe de z et de $\bar{\zeta}$ et se prolonge dans $\mathfrak{H}(G_z)\times\mathfrak{H}(\bar{G}_\zeta)$ comme fonction holomorphe de $(z, \bar{\zeta})$. Si $G\subset G_0$, et si $K_G(z)=K_G(z, \bar{z})$ se prolonge dans G_0 comme fonction analytique réelle, toute $f\in L^2(G)$ ainsi que $K_G(z, \bar{\zeta})$ se prolongent analytiquement (respectivement dans G_0 et dans $G_0\times\bar{G}_0$; le rôle de $K_G(z, \zeta)$ comme noyau reproduisant s'étend à $\mathfrak{G}_0$; de même la métrique de S. Bergmann [cf. aussi H. J. Bremermann, Lectures on functions of a complex variable, Univ. of Michigan Press, 1955, pp. 349–383; MR **17**, 529]. *P. Lelong* (Paris).

Grauert, Hans. Charakterisierung der Holomorphiegebiete durch die vollständige Kählersche Metrik. Math. Ann. **131** (1956), 38–75.

Sur une variété de Stein V^n on peut définir une métrique kaehlerienne à partir de fonctions f_k holomorphes sur V^n: on construit $R=\sum_1^\infty a_k f_k \bar{f}_k$, de manière que la série converge uniformément sur tout compact de V^n; R est fonction plurisousharmonique sur V^n; la métrique $ds^2=\sum_{\alpha,\beta}(\partial^2 R/\partial z_\alpha\partial\bar{z}_\beta)dz_\alpha d\bar{z}_\beta$ est évidemment kaehlerienne; on peut la construire de manière qu'elle soit définie positive et qu'elle soit complète sur V^n, c'est à dire que la distance $d(x_0, x_\nu)\to+\infty$ pour toute suite infinie x_ν sans point d'accumulation sur V^n, x_0 étant donné arbitrairement sur V^n.

La construction précédente fournit encore une métrique kaehlerienne positive complète sur un variété V'^n déduite de V^n par suppression des points d'une sous-variété analytique A; si dim $A\leq n-2$, ceci montre que l'existence d'une métrique kaehlerienne complète sur V'^n n'entraîne pas que V'^n soit holomorphe-convexe. Pour un domaine de Reinhardt K^n, de centre l'origine, une telle métrique supposée à coefficients analytiques réels existe si et seulement si K^n ne diffère d'un domaine de Reinhardt holomorphe-convexe que par suppression de variétés axiales $[z_{i_1}=0, \cdots, z_{i_q}=0]$ de dimension complexe $n-q$, avec $q\geq 2$.

Si un domaine $\mathfrak{G}$ de C^n possède une métrique kaehlerienne complète, le théorème du disque lui est applicable sous la forme suivante: $\Gamma(G)$ étant un disque obtenu comme image du cercle $|u|\leq 1$ par des fonctions $z_j=f_j(u, t)$, holomorphes pour $|u|\leq 1$, continues pour $|u|\leq 1$, $0\leq t\leq 1$, si $\Gamma(t)$ intérieur à $\mathfrak{G}$ pour $0\leq t<1$, si le bord $b(t)$, image de $|u|=1$ est intérieure à $\mathfrak{G}$ pour $0\leq t\leq 1$, alors $\Gamma(1)$ ne peut avoir avec la frontière de $\mathfrak{G}$ un continu de points communs. Si un domaine de Hartogs, $\zeta=(z_2, \cdots, z_n)\in d$, $|z_1|<R(\zeta)$, a une métrique de ce type, si R est deux fois dérivable, $-\log R(\zeta)$ est plurisousharmonique. Ceci conduit à l'enoncé: un domaine sans ramification intérieure, dont chaque point possède un voisinage qui peut être représenté biunivoquement sur C^n, et qui possède une métrique du type indiqué, est nécessairement pseudo-convexe si sa frontière est suffisamment régulière (l'auteur la suppose analytique réelle). En invoquant le théorème de K. Oka [Jap. J. Math. **23** (1953), 97–1955; MR **17**, 82] on en déduit qu'un tel domaine est domaine d'holomorphie et est holomorphe-convexe. Le mémoire contient encore les énoncés suivants: sur une variété de Stein V^n, on sait qu'un ensemble analytique peut être défini en annulant des fonctions holomorphes sur V^n; on établit ici qu'on peut le définir en annulant $n+1$ fonctions au plus. Si dans un domaine de C^n on a défini une métrique hermitienne Λ, et si les variétés analytiques à une dimension complexe sont des surfaces minima pour Λ, la métrique Λ est kaehlerienne. *P. Lelong* (Paris).

See also: Martin-Chern-Zariski, p. 1129; Shimoda, p. 1114.

Harmonic Functions, Potential Theory

Kuramochi, Zenjiro. Evans's theorem on abstract Riemann surfaces with null-boundaries. I, II. Proc. Japan Acad. **32** (1956), 1–6, 7–9.

This paper is concerned with the existence of a positive harmonic function on a parabolic Riemann surface less (say) a disk which vanishes continuously on the frontier of the disk and becomes infinite at each point of a closed (in the sense of Martin) subset of the ideal boundary. The argument employed is based on the Martin theory and a generalization of the notion of transfinite diameter expressed in terms of Green's functions and their limit functions. The reviewer would have welcomed more details concerning the generalized transfinite diameter. *M. Heins* (Princeton, N.J.).

Pfluger, Albert. Ein alternierendes Verfahren auf Riemannschen Flächen. Comment. Math. Helv. **30** (1956), 265–274.

Three potential-theoretic existence questions for Riemann surfaces are treated by an alternating procedure which differs from the classical alternating procedure in that it shifts attention to the differentials of the functions constructed by the procedure. The convergence of the sequence of differentials is treated by Hilbert space methods with the aid of the Dirichlet bilinear form. The problems considered are: (1) the existence of a harmonic function with prescribed period across a given nonseparating retrosection, (2) the representation of an exact differential as the sum of a harmonic differential and a total differential, (3) the Neumann alternating procedure. *M. Heins* (Princeton, N.J.).

Pfluger, Albert. Ein Approximationssatz für harmonische Funktionen auf Riemannschen Flächen. Ann. Acad. Sci. Fenn. Ser. A. I. no. **216** (1956), 8 pp.

The analogue for harmonic functions of the Behnke-Stein approximation theorem is established. A Runge-type argument is employed but the replacement of analyticity by harmonicity calls for an approach which differs entirely from that of the analytic case. Applications are made to the construction of harmonic functions with a countable set of assigned isolated singularities and the Hodge decomposition of linear differential forms on a Riemann surface (and to the existence of harmonic differential forms with prescribed periods). *M. H. Heins.*

Choquet, Gustave; et Deny, Jacques. Aspects linéaires de la théorie du potentiel. I. Etude des modèles finis. C. R. Acad. Sci. Paris **242** (1956), 222–225.

With this paper the authors begin a study of potential theory in locally compact spaces in order to obtain the logical relations that exist between the various concepts in the classical theory.

In this paper the authors consider spaces E that consist of

n points. A kernel G is a real valued function $G(i, j) \geqq 0$, for $1 \leqq i, j \leqq n$. A measure μ is an n-tuple of real numbers $m_1, \cdots, m_n$; the support S_μ of μ consists of those points i for which $m_i \neq 0$. The potential due to the measure μ is defined by $G\mu(i) = \sum_{j=1}^n G(i, j)m_j$. By interpreting μ and $G\mu$ as vectors in n-space, $\mathbf{\mu}$ and $G\mathbf{\mu}$, respectively, the authors note that $G\mathbf{\mu}$ is the transform of $\mathbf{\mu}$ by the matrix $\|G(i, j)\|$. The authors proceed to study various notions such as balayage, domination (for λ and μ positive, the relation $G\mu \geqq G\lambda$ on the support S_λ implies $G\mu \geqq G\lambda$ throughout), and the principle of the lower bound (for λ and μ positive, there always exists a positive ν such that $G\nu = \inf [G\lambda, G\mu]$.) The authors introduce an idea of regularity, and an idea of duality. They show for example that balayage is dual to the principle of domination. The authors include a special study of degenerate and non-degenerate kernels as well as one of symmetric kernels. In this latter connection they show that when G is symmetric, the domination principle and the balayage principle are equivalent; moreover, G is of positive type, i.e., for μ arbitrary, the energy $I(\mu) = \sum_{i,j} [G(i, j)m_i m_j]$ is never negative. The equilibrium principle is intentionally left for a subsequent study. *M. Reade.*

Naïm, Linda. Sur l'allure des fonctions surharmoniques positives à la frontière de Martin. C. R. Acad. Sci. Paris **241** (1955), 1907–1910.

Let Ω be a Green space and let Δ be its Martin boundary [Brelot and Choquet, Ann. Inst. Fourier, Grenoble 3 (1951), 199–263; MR **16**, 34]. Then for P_0 fixed, the ratio of Green's functions $G(M, P)/G(M, P_0)$ gives rise to a harmonic kernel $K(M, P)$ for $M \in \hat{\Omega}$. Similarly, the ratio $K(M, P)/G(P, P_0)$ gives rise to a "super-harmonic" kernel denoted by $\theta(Q, M)$. Now for each distribution $\mu(P)$ of positive mass on $\hat{\Omega} - \{P_0\}$, the author defines θ-potentials $\int \theta(M, P)d\mu(P)$. In this paper, the author studies the properties of θ and of θ-potentials. She also studies minimal points as well as conditions that a set $E \subset \Omega$ be éffilé at a point. The author also studies the boundary behavior of functions that are superharmonic in Ω. Her results extend certain earlier results due to Lelong-Ferrand [Ann. Sci. École Norm. Sup. (3) **66** (1949), 125–159; MR **11**, 176]. *M. Reade.*

Naïm, Linda. Etude et applications de la notion d'effilement à la frontière de Martin. C. R. Acad. Sci. Paris **242** (1956), 1107–1110.

The author continues her study [see the paper reviewed above] of the éffilement of a set in a Green space. A new condition allows her to make applications to the study of the principle of the maximum and to the Dirichlet problem. She makes use of results due to Brelot [same C.R. **240** (1955), 142–144; MR **16**, 923; see also the paper reviewed below]. *M. Reade* (Ann Arbor, Mich.).

Brelot, Marcel. Etude axiomatique du problème de Dirichlet. C. R. Acad. Sci. Paris **242** (1956), 327–329.

The author continues the study of the axiomatics of the Dirichlet problem in a Green space Ω. Let $\check{\Omega}$ be the completion of the metric Green space Ω, and let h be a fixed position function harmonic in Ω. Now let f be a real-valued function defined on the boundary $\Gamma = \check{\Omega} - \Omega$ and let u be subharmonic (or $-\infty$) in Ω such that (i) u/h is bounded above, and (ii) $\lim\sup (u/h) \leqq f$ at each point of Γ. Then the envelopes $H_{f,h} = \sup [u|u \text{ above}]$ and $\bar{H}_{f,h} = \inf[u|u \text{ above}]$ are either $-\infty$, $+\infty$ or harmonic. Now by considering two axioms relative to the points of Γ, the author generalizes certain earlier results [Brelot and Choquet, Ann. Inst. Fourier, Grenoble 3 (1951), 119–263; MR **16**, 34]. The author also studies the particular case when Ω is such as to be everywhere dense in a metric compactification $\bar{\Omega}$; in this case a single axiom relative to the equality of the envelopes is used. A preliminary study of the relation between the two axiom systems is made. The author includes a function-theoretic application as well as applications to the Martin frontier of Ω. *M. Reade* (Ann Arbor, Mich.).

Naïm, Linda. Propriétés et applications de la frontière de R. S. Martin. C. R. Acad. Sci. Paris **242** (1956), 2695–2698.

The author uses some of the ideas of the paper reviewed above to continue the study of the relation between the axiom systems noted in the preceding review. She also studies minimal harmonic functions, and their poles, as well as the boundary behavior of the envelopes noted above in the case when they are identical; this is done in the general setting introduced by Brelot in the paper reviewed above. *M. Reade* (Ann Arbor, Mich.).

Hayman, W. K. Questions of regularity connected with the Phragmén-Lindelöf principle. J. Math. Pures Appl. (9) **35** (1956), 115–126.

Let u be subharmonic ($\not\equiv -\infty$) in $\Re z > 0$ and satisfy the Phragmén-Lindelöf boundary condition

$$\limsup_{z \to i\eta} u(z) \leqq 0$$

for all finite real η. Let $\alpha = \sup_{\Re z > 0} u(z)/\Re z < +\infty$. The author ameliorates a result of Ahlfors and Heins [Ann. of Math. (2) **50** (1949), 341–346; MR **10**, 522] by showing that there exists an r-set Δ_0 of finite logarithmic length such that $\lim_{r \to \infty} u(re^{i\theta})/r = \alpha \cos \theta$ holds uniformly for $|\theta| < \frac{1}{2}\pi$ outside Δ_0. The proof is based upon an ingenious modification of the representation given by the Riesz decomposition theorem for the subharmonic functions under consideration. Related results are also given. *M. H. Heins* (Princeton, N.J.).

Smolickiĭ, H. L. An estimate for the derivatives of the Neumann function. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 785–787. (Russian)

The boundedness of $r_{xy}^{m+1}|\partial^m G(x, y)/\partial x_1^{m_1}\partial x_2^{m_2}\partial x_3^{m_3}|$, known for the case of G the Green's function for the Laplace operator, is here established (i) for G the Neumann function and (ii) for G the Green's function for the mixed boundary-value problem. Certain smoothness properties are required of the boundary. *M. G. Arsove.*

Rényi, A.; und Rényi, K. Über die Schlichtheit des komplexen Potentials. I. Magyar Tud. Akad. Alkalm. Mat. Int. Közl. 3 (1954), 353–367 (1955). (Hungarian. Russian and German summaries)

Let T be a simply connected domain in the complex z-plane with $z = \infty$ as a simple boundary point. Let $w = f(z) = \varphi + i\psi$ be the complex potential of a velocity field in T. The author obtains for the univalency of the mapping $w = f(z)$ of T onto $Iw > 0$ the following necessary and sufficient conditions: The flux across any curve joining a boundary point with an interior point of T is well defined and different from zero. A similar theorem holds for domains conformally equivalent to a strip. In addition to the condition mentioned it must then be assumed that the flux across any curve joining two

boundary points of different boundary components is constant. *G. Szegö* (Stanford, Calif.).

Kondurar', V. T. On the potential of the mutual attraction of two nonhomogeneous rigid bodies. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **5** (1954), 103–115. (Russian)

In the theory of motion of two ellipsoids the need arises for representation of the potential of the mutual attraction of the two bodies by an infinite series.

In some previous papers [Moskov. Gos. Univ. Trudy Gos. Astr. Inst. **21** (1952), 115–134, 135–158], which have not been available to the reviewer, the author claims to have given such representations for the case of two homogeneous ellipsoids. In the present paper an analogous expansion is given for the case of two nonhomogeneous ellipsoids. This expansion assumes a particularly simple form under the assumption that inside each ellipsoid the surfaces of equal density are concentric and similar to the exterior surfaces of the two ellipsoids.

Since the general case of motion, when the mutual disposition of the two ellipsoids is arbitrary, is extremely complicated, the author considers the particular case when the major axes of the two ellipsoids of revolution lie in the same line.

The expansion of the potential is given in the form of a double infinite series in powers of the reciprocal r^{-1} of the distance between the centers of the two ellipsoids and powers of the quantities l_1 and l_2, where

$$l_1=(c_1^2/a_1^2-1)^{\frac{1}{2}},\quad l_2=(c_2^2/a_2^2-1)^{\frac{1}{2}};$$

c_1, c_2 denote the semi-major axes of the two ellipsoids and $a_1=b_1$, $a_2=b_2$ the semi-minor axes, respectively. The coefficients depend upon the dimensions of the two ellipsoids and upon the distributions of density inside the two ellipsoids. *E. Leimanis* (Vancouver, B.C.).

Szegö, G. On a certain kind of symmetrization and its applications. Ann. Mat. Pura Appl. (4) **40** (1955), 113–119.

Let C denote a smooth Jordan curve in the plane, star-shaped with respect to an interior point O. Let $r=r(\varphi)$ be the polar equation of C with origin O. Definition: The symmetrization S_n ($n\geqq 2$, integer), applied to C and O, replaces C by the curve $\bar{C}$:

$$\varrho(\varphi)=[r(\varphi)r(\varphi+2\pi/n)\cdots r(\varphi+(n-1)2\pi/n)]^{1/n}.$$

Theorem: Let the cross-section D of a cylindrical condenser be bounded by the curves L_1 and L_0 (L_1 enclosing L_0), both being star-shaped with respect to O (point in the interior of L_0). Then the logarithmic capacity of D is not smaller than that of the cross-section $\bar{D}$ which arises by application of S_n to L_1 and L_0. As limiting cases [cf. Pólya and Szegö, Isoperimetric inequalities in mathematical physics, Princeton, 1951, pp. 44–45; MR **13**, 270] corresponding statements concerning the inner radius (with respect to an interior point) and the outer radius (transfinite diameter) of a closed curve are obtained. The author concludes with extensions to higher-dimensional spaces. Theorem: Let A be a smooth closed surface in 3-space, star-shaped with respect to O. Choose a number n ($\geqq 2$, integer) and a unit vector **a** arbitrarily. The following process of symmetrization does not increase the capacity of A: Intersect A by a variable cone of vertex O and axis **a**. Consider any n points of the curve of intersection which correspond to n regularly spaced values of the meridian angle of the cone, and replace their distance from O by their arithmetic mean. *A. Huber* (Zurich).

See also: Brousse, p. 1090; Rothe and Schmeidler, p. 1061.

Series, Summability

Have German & English

★ **Knopp, Konrad. Szeregi nieskończone. [Infinite series.]** Państwowe Wydawnictwo Naukowe, Warszawa, 1956. 608 pp. zł. 48.60.

Translation of the author's Theorie und Anwendung der unendlichen Reihen [4th ed., Springer, Berlin, 1947; MR **10**, 446].

Matsuyama, Noboru; and Takahashi, Shigeru. On some property of a gap sequence. Sci. Rep. Kanazawa Univ. **3** (1955), no. 1, 27–34.

Sharpening previous results of Fortet [Studia Math. **9** (1940), 54–70; MR **3**, 169], Maruyama [Kodai Math. Sem. Rep. **1950**, 31–32; MR **12**, 406] and Izumi [J. Math. Tokyo **1** (1951), 1–22; MR **14**, 553]. The authors prove the following theorem:

Let $f(t)$ be a bounded Borel measurable function of period 1 for which $\int_0^1 f(t)dt=0$ and which belongs to $\mathrm{Lip}^*(1,\alpha)$ (i.e. $\int_0^1 \max_{0<h\leq u}|f(x+h)-f(x)|dx=O(u^\alpha)$). Let further φ_k ($k=1, 2, \cdots$) be an increasing sequence of positive numbers. Put

$$\sigma^2=\lim_{n\to\infty} n^{-1}\int_0^1\Bigl(\sum_{k=1}^n f(2^k t)\Bigr)^2 dt.$$

Then the necessary and sufficient condition that for almost all t there should exist infinitely many n for which

$$\sum_{k=1}^n f(2^k t)>\sigma n^{\frac{1}{2}}\varphi_n$$

is that the series

$$\sum \varphi_k k^{-1}\exp\{-\tfrac{1}{2}\varphi_k^2\}$$

should diverge.

The principal tool of the authors is the theorem of Feller [Trans. Amer. Math. Soc. **54** (1943), 373–402; MR **5**, 125]. *P. Erdős.*

Clunie, J. Series of positive terms. J. Univ. Bombay. Sect. A. (N.S.) **24** (1955), no. 38, 10–12.

Let $f(x)$ be a positive function which has bounded variation over $1\leqq x<\infty$, and let the total variation of $f(x)$ over the interval $\alpha\leqq x<\infty$ be denoted by $\mathrm{TV}_\alpha^\infty f(x)$. It is shown that if $\mathrm{TV}_\alpha^\infty f(x)\leqq\phi(\alpha)$, where $\phi(\alpha)$ is a positive monotone decreasing function for which $\sum_{n=1}^\infty \phi(n)<\infty$, then the two series $\sum f(n)$ and $\sum 2^n f(2^n)$ are both convergent or both divergent. *R. P. Agnew.*

Lotockiĭ, A. V. On a linear transformation of sequences and series. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki **4** (1953), 61–91. (Russian)

The author introduces a triangular matrix method for evaluation of series and sequences which seems to be new and exceptionally useful and interesting. For each $n=1, 2, 3, \cdots$, let constants $p_{n1}, p_{n2}, \cdots, p_{nn}$ be defined by the identity

$$\sum_{k=1}^n p_{nk}x^k=x(x+1)(x+2)\cdots(x+n-1).$$

A series $u_1+u_2+\cdots$ and its sequence $s_1, s_2, \cdots$ of partial sums is evaluable to σ by the Lotockiĭ method L if

$\sigma_n \to \sigma$ as $n \to \infty$, where

$$\sigma_n = \sum_{k=1}^{n} (p_{n,k}/n!) s_k.$$

It is shown that L is regular, and L is applied to evaluate several specific numerical series. The series-to-series version of L and the inverse of L are worked out, and necessary conditions for evaluability L are obtained. It is shown that L includes the Euler method E and that L is consistent with the Borel exponential and integral methods. The paper contains a considerable number of misprints and other errors which are not hard to rectify. The reviewer has found that the results of the author can be substantially extended in many ways. For example, the author's proof that the series $0+1+z+z^2+\cdots$ is evaluable L to $1/(1-z)$ when z is a negative integer can be replaced by a proof that the geometric series $1+z+z^2+\cdots$ is evaluable L to $1/(1-z)$ over the whole half-plane $\mathrm{Re}\, z<1$. *R. P. Agnew* (Ithaca, N.Y.).

Kangro, G. On summation of infinite series by matrix methods. Tartu. Gos. Univ. Trudy Estest.-Mat. Fak. 37 (1955), 150–190. (Russian. Estonian summary)

When a given series $u_0+u_1+\cdots$ is evaluable to s by a given method A, this value s is denoted by the operator symbol $A\{\sum u_k\}$. When A and B are two given methods, there is the question whether the formula

$$\lim_{\theta\to 1} B\{\sum u_k\theta^k\{=A\{\sum u_k\}$$

is valid whenever the right member exists. The author treats this question and other questions involving the termwise derivatives and integrals of a power series and the Cauchy product of two series. Supposing that A and B are methods determined by triangular matrices and that A has an inverse, he gives answers to several of these questions in terms of the elements of the matrices A and B and the elements of other matrices related to A and B. Most of the paper is devoted to the case in which A and B are Riesz methods (R, p_n) and (R, q_n). There are 23 theorems, many of which have hypotheses and conclusions involving absolute convergence and absolute evaluability by Riesz methods. *R. P. Agnew* (Ithaca, N.Y.).

Volkov, I. I. Some questions concerning linear matrix transformations. Dokl. Akad. Nauk SSSR (N.S.) 106 (1956), 591–594. (Russian)

In the theory of linear matrix transformations of sequences, it is the usual procedure to consider only sequences $s_0, s_1, s_2, \cdots$ for which the series in $\sigma_n=\sum_{k=0}^{\infty} a_{nk}s_k$ converges when $n=0, 1, 2, \cdots$, or at least when n is sufficiently great. In cases where, for each n, $a_{nk}\neq 0$ for an infinite set of values of k this restriction is nontrivial and it is the purpose of the present paper to remove it. Problems involving regularity, total regularity, equivalence, and Knopp's core (Kern) are formulated in terms of the function

$$\limsup_{n\to\infty}\, \limsup_{q\to\infty}\, \mathrm{Re}\, e^{i\phi} \sum_{k=0}^{q} a_{nk}s_k,$$

where ϕ is real. Some results involving these concepts are given without proofs. *R. P. Agnew* (Ithaca, N.Y.).

Kangro, G. On summability factors. Tartu. Gos. Univ. Trudy Estest.-Mat. Fak. 37 (1955), 191–232. (Russian. Estonian summary)

Let B be a series to sequence transformation and let A be a series to sequence (or sequence to sequence) transformation. Two classes P and Q of complex sequences then determine the problem of characterizing the class $F(A, B, P, Q)$ of factor sequences $\varepsilon_0, \varepsilon_1, \varepsilon_2, \cdots$ such that the B transform of the series $\varepsilon_0u_0+\varepsilon_1u_1+\cdots$ (or of the series $\varepsilon_0s_0+\varepsilon_1s_1+\cdots$) belongs to class Q whenever the A transform of the series $u_0+u_1+\cdots$ (or sequence $s_0, s_1, \cdots$) belongs to class P. Numerous references are given to papers where this problem is solved for special choices of A, B, P and Q. The class P is taken to be either (1) the class of convergent sequences or (2) the class of absolutely convergent sequences, that is, the class of sequences having bounded variation or (3) the class of bounded sequences; and the class Q is taken to be the same or a different one of these three classes. It is usually assumed that A and B are regular or conservative (convergence preserving) matrix transformations and that A has an inverse. The class $F(A, B, P, Q)$ of factor sequences is determined in many cases, and in some of these cases it is shown what the results reduce to when $A=B$ and when A is a Riesz transformation. *R. P. Agnew.*

Davydov, N. A. On a property of the methods of Cesàro for summation of series. Mat. Sb. N. S. 38(80) (1956), 509–524. (Russian)

Let $p>-1$. Let $S_0, S_1, \cdots$ be a complex sequence evaluable to S by the Cesàro method C_r of order r. Let G be a closed convex set in the complex plane and, for each $\varepsilon>0$, let G_ε denote the set of complex numbers z_ε such that $|z_\varepsilon - z|\leq\varepsilon$ for at least one z in G. Suppose that for each $\varepsilon>0$ there exists a number λ and two sequences m_k and n_k of integers such that $\lambda>1$, $n_k\to\infty$, $m_k/n_k\geq\lambda$, and S_j belongs to the set G_ε whenever $n_k\leq j\leq m_k$. Then S belongs to the set G. This theorem implies several Tauberian theorems which show that if a sequence $S_0, S_1, \cdots$ is evaluable C_p to S and if some (but not necessarily all) of the numbers $S_0, S_1, \cdots$ satisfy an appropriate Tauberian condition, then one or more subsequences of the sequence $S_0, S_1, \cdots$ must converge to S. Similar results are obtained for those Riesz methods for which the transform $\sigma_0, \sigma_1, \cdots$ of a sequence $S_0, S_1, \cdots$ is defined by

$$\sigma_n = \frac{p_0S_0+p_1S_1+\cdots+p_nS_n}{p_0+p_1+\cdots+p_n},$$

where $p_0>0$, $p_k\geq 0$, and $\sum_{k=0}^{\infty} p_k=\infty$. *R. P. Agnew.*

Fedulov, V. S. On (C, 1, 1)-summability of a double orthogonal series. Ukrain. Mat. Ž. 7 (1955), 433–442. (Russian)

Let $\phi_{i,j}(x, y)$ be a set of functions orthonormal over the rectangle R defined by $a\leq x\leq b$, $c\leq y\leq d$. Let $S_{m,n}(x, y)$ and $\sigma_{m,n}(x, y)$ denote the sequence of partial sums and the $(C, 1, 1)$ transform of the double orthogonal series (*) $\sum a_{ij}\phi_{ij}(x, y)$ in which the coefficients a_{ij} are real constants. It is shown that if $\sum a_{ij}^2<\infty$, then $\sigma_{m,n}(x, y)$ converges almost everywhere (a.e.) over R as $m, n\to\infty$ if and only if $S_{2^m,2^n}(x, y)$ converges a.e. over R as $m,n\to\infty$. This result is used to prove that if

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{m,n}^2[\log\log(m+3)]^2[\log\log(n+3)]^2<\infty,$$

then (*) is evaluable $(C, 1, 1)$ a.e. over R, and it is shown that the product of the factors involving the logarithms cannot be replaced by a function $w(m, n)$ of lower order. Finally the orthogonal series are restricted to trigonometric Fourier series of functions in L_2, and two theorems on restricted convergence of $\sigma_{m,n}(x, y)$ and of lacunary subsequences of $S_{m,n}(x, y)$ are obtained. *R. P. Agnew.*

Ostrowski, A. Sur les critères de convergence et divergence dus à V. Ermakof. Enseignement Math. (2) 1 (1956), 224–257.

The simplest form of Ermakof's convergence-divergence criterion [Bull. Sci. Math. Astr. **2** (1871), 250–256; **7** (1883), 142–144; Knopp, Theorie und Anwendungen der unendliche Reihen, 4th ed., Springer, Berlin, 1948, pp. 305, 306, MR **10**, 446] states that if for $x \geqq a$, $f(x)$ is positive and continuous, $\psi(x)$ is positive, has a positive and continuous derivative and satisfies the inequality $\psi(x) > x$, then $\int_a^\infty f(x)dx$ converges provided there exists a q, $0<q<1$, such that $f(\psi(x))\psi'(x) \leqq qf(x)$ and is divergent if $f(\psi(x)\psi'(x) \geqq f(x)$, from which one derives obvious conditions for the convergence or divergence of $\sum f(n)$, if $f(x)$ is decreasing. In his second paper Ermakof derives a criterion for the convergence properties of the series $\sum f(n)$ without the intermediation of the integral $\int f(x)dx$, so that the monotoneity of $f(x)$ may be dropped. The present note sets up rigorous conditions for the validity of Ermakof's reasoning, and proves that if in addition beginning with some x either (1) $\psi'(x)$ does not decrease and attains or exceeds unity, or (2) does not increase, then the convergence criterion holds if $f(x)$ is positive measurable, and bounded in every finite subinterval, while the divergence conditions require that $f(x)$ be positive measurable and $1/f(x)$ be bounded on every finite subinterval. Following Ermakof the proof is made to depend on the existence of an increasing function $\phi(x)$ satisfying the Abel's functional equation $\psi(\phi(x)) = \phi(x)+1$, for which $\sum \phi_+'(n)$ is divergent and $\sum \phi_+'(n)/(\phi(n))^2$ is convergent. Some statements of Ermakof's first paper are also shown to be untenable via counter examples. *T. H. Hildebrandt.*

Hirokawa, Hiroshi. Riemann-Cesàro methods of summability. Tôhoku Math. J. (2) 7 (1955), 279–295.

Soit p un nombre entier positif. La série (1) $\sum_1^\infty a_k$ est sommable par le procédé Riemann-Cesàro d'ordre p et de l'indice α, sommable - (R, p, α) vers s, si

$$\sigma(p, \alpha, t) = C_{p,\alpha}^{-1} t^{\alpha+1} \sum_{k=1}^{\infty} \left(\frac{\sin kt}{kt}\right)^p s_k^\alpha$$

converge dans $0<t<t_0$ et si $\sigma(p, \alpha, t) \to s$ avec $t \to 0$, où

$$C_{p,\alpha} = [1/\Gamma(\alpha+1)] \int_0^\infty u^{\alpha-p} (\sin u)^p du \quad (-1<\alpha<p-1),$$
$$C_{1,0} = \pi/2, \quad C_{p,-1} = 1.$$

Alors les résultats suivants sont demontrés. Soit (1) $(C, p-\delta)$-sommable, $0<\delta<1$ vers s et $\sum_1^n |\sigma_k^{p-\delta-1}| = O(n)$, alors (1) est sommable-(R, p, α) vers s, pour $-1 \leqq \alpha < p-\delta-1$. Le procédé $(R, 1, \alpha)$, $-1 \leqq \alpha \leqq 0$, n'est pas régulier. Un résultat semblable a lieu si (1) est sommable-$|C, p|$. Si (2) $\sum_n^{2n} (|a_k| - a_k) = O(1)$ et si (1) est sommable-Abel vers s, alors (1) est de même sommable-$(R, 1, \alpha)$ vers s pour $-1 \leqq \alpha \leqq 0$. Si (2) est d'ordre n^{1-r}, $0<r<1$ et si $\sum_1^n |s_n - s| = o(n/\log n)$, alors (1) est $(R, 1, \alpha)$-sommable vers s pour $-1 \leqq \alpha \leqq 0$. *M. Tomić* (Beograd).

Peyerimhoff, Alexander. Über Summierbarkeitsfaktoren und verwandte Fragen bei Cesàroverfahren. I. Acad. Serbe Sci. Publ. Inst. Math. 8 (1955), 139–156.

This paper and its second part are intended to present a unified exposition of theorems about the (C_α, C_β), $(C_\alpha, |C_\beta|)$ and the $(|C_\alpha|, |C_\beta|)$ summability factors. For example, the factors ε_n of the class $(A, |B|)$ are defined by the property to transform each A-summable series $\sum a_n$ into a $|B|$-summable series $\sum \varepsilon_n a_n$. If the matrix A is normal (i.e. triangular and $a_{nn} \neq 0$ for all n), it is easy to characterize the (A, B) and other types of methods by means of general theorems. The main difficulty is to transform these conditions into simpler ones when A, B are Cesàro methods. The following results are obtained. For $\varepsilon_n \in (C_\alpha, C_\beta)$ $(\alpha, \beta \geqq 0)$ it is necessary and sufficient that $\varepsilon_n = O(1)$ and that (a) $\delta_n \to 0$, $\sum n^\beta |\Delta^{\beta+1} \delta_n| < +\infty$ should imply (b) $\sum n^\alpha |\Delta^{\alpha+1}(\varepsilon_n \delta_n)| < +\infty$. For $\varepsilon_n \in (C_\alpha, |C_\beta|)$, the conditions are $\varepsilon_n = O(1)$ and (a′) $\delta_n = O(1)$, $\Delta^\beta(\delta_n/n) = O(n^{-\beta-1})$ implies (b), and for $\varepsilon_n \in (|C_\alpha|, |C_\beta|)$, $\varepsilon_n = O(1)$ and (a″) $\delta_n \to 0$, $\Delta^\beta \delta_n = O(n^{-\beta-1})$ implies $\Delta^\alpha(\varepsilon_n \delta_n) = O(n^{-\alpha-1})$.
G. G. Lorentz (Detroit, Mich.).

Mohanty, R.; and Mahapatra, S. On the absolute logarithmic summability of a Fourier series and its differentiated series. Proc. Amer. Math. Soc. 7 (1956), 254–259.

The series $\sum a_n$ is said to be $|R, \log n, 2|$ summable, if, for an $A>0$,

$$\int_A^\infty \frac{1}{w(\log w)^3} \left| \sum_{n<w} \log n \log \frac{w}{n} a_n \right| dw < \infty.$$

The authors prove that (i) if $\log(2\pi/t) \int_t^\pi (\varphi_x(u)/u) du$ is of bounded variation in the interval $(0, \pi)$, then the Fourier series of $f(t)$ is $|R, \log n, 2|$ summable at $t=x$, where $\varphi_x(u) = f(x+u) + f(x-u) - 2f(x)$; and (ii) if

$$\log(2\pi/t) \int_t^\pi [f(x+u) - f(x-u)] u^{-2} du$$

is of bounded variation in the interval $(0, \pi)$, then the derived Fourier series of $f(t)$ is $|R, \log n, 2|$ summable at $t=x$. *S. Izumi* (Sapporo).

Singh, Vikramaditya; and Thron, W. J. A family of best twin convergence regions for continued fractions. Proc. Amer. Math. Soc. 7 (1956), 277–282.

Let regions E_1 and E_2 be defined by $|z| \leqq \varrho$ and $|z \pm i| \geqq \varrho + \varepsilon$, respectively, where $0<\varrho<1$, and $\varepsilon>0$. Thron [Duke Math. J. **10** (1943), 677–685; MR **5**, 118] showed that if $c_{2n-1} \in E_1$, $c_{2n} \in E_2$ $(n=1, 2, \cdots)$, then the continued fraction $1 + c_1^2/1 + c_2^2/1 + \cdots$ converges. He also showed that in case the theorem holds for $\varepsilon=0$, the regions E_1 and E_2 are a set of best twin convergence regions. The present paper gives a proof of the theorem with $\varepsilon=0$. The proof depends on the familiar "nest of circles" technique. *W. T. Scott* (Evanston, Ill.).

See also: Zygmund, p. 1080.

Interpolation, Approximation, Orthogonal Functions

Pflanz, E. Zur Berechnung der Werte eines Polynomes mit dem Hornerschen Verfahren. Z. Angew. Math. Mech. 36 (1956), 152.

Inozemcev, O. I.; and Marčenko, V. A. On majorants of genus zero. Uspehi Mat. Nauk (N.S.) 11 (1956), 173–178. (Russian)

A function $\omega(x)$ is called a strong majorant of $\alpha(x)$ $(-\infty<x<\infty)$, if for every $\lambda>0$ $|\alpha(x)/\omega(\lambda x)|$ is bounded. In the theory of the Bernstein approximation problem it is of some interest to know whether a function $\alpha(x)$ has

a strong majorant in the class of entire functions of the form $\prod_{n=1}^{\infty}(1+ic_n z)$, c_n real. The authors show that this is the case if and only if

$$\int_0^{\infty}\log(\sup_{|t|\leq x}(1+|\alpha(t)|)/(1+x^2)dx<\infty.$$

They also show that this condition is satisfied, if $\alpha(x)\geqq 1$, $\alpha(x+y)\leqq c\alpha(x)\alpha(y)$ and $\int_0^{\infty}\log\alpha(x)/(1+x^2)dx<\infty$.

W. H. J. Fuchs (Ithaca, N.Y.).

Schöbe, Waldemar. Rationale Approximationen der Potenzfunktion. Bl. Deutsch. Ges. Versicherungsmath. 2 (1956), 469–484.

The function x^y, where x is positive and y is real, is approximated by a rational function

$$f_k(x, y)=H_k(x, y)/H_k(x, -y),$$

where $H_k(x, y)=\sum_{m=0}^{k}\binom{k+y}{m}\binom{k-y}{k-m}x^m$, and it is shown that $\lim_{k\to\infty}f_k(x, y)=x^y$. The approximation gives the exact value for $x=1$, and for $y=0, \pm1, \pm2, \cdots, \pm k$, and has some properties of the original function: e.g., the reciprocal is obtained on either replacing x by $1/x$ or y by $-y$. The polynomials $H_k(x, y)$ satisfy the recurrence relation:

$$k(k+1)H_{k+1}=k(2k+1)(x+1)H_k-(k^2-y^2)(x-1)^2H_{k-1}.$$

If $L_k(x, y)$ is defined as $[H_k(x, y)-H_k(x, -y)]y^{-1}$, it is shown that $\lim_{k\to\infty}[L_k(x, 0)/H_k(x, 0)]=\log x$; and further that $\lim_{k\to\infty}[P_k(z)/P_k(-z)]=e^z$, where

$$P_k(z)=\sum_{m=0}^{k}\binom{2k-m}{k}z^m/m!$$

[Note: the rational approximation to x^y is the same as that given by Thiele's continued fraction approximation based on reciprocal differences, using the arguments $y=0, \pm1, \pm2, \cdots, \pm k$. See Milne-Thomson, The calculus of finite differences, Macmillan, London, 1933; Thiele, Interpolationsrechnung, Teubner, Leipzig, 1909.]

T. N. E. Greville (Washington, D.C.).

Selivanova, S. G. Asymptotic estimates of approximations of differentiable nonperiodic functions by Čebyšev sums. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 648–651. (Russian)

Soit $W^{(r)}$ la classe des fonctions $f(x)$ définies sur $[-1, +1]$ r fois dérivables, avec $|f^{(r)}(x)|\leqq 1$; soit $U(f; x)$ la $n$ième somme partielle du développement de $f(x)$ en polynômes de Tchebitcheff, et $E_n=\sup|f(x)-U_n(f; x)|$ quand $f\in W^{(r)}$. L'auteur montre que

$$E_n=4\pi^{-2}n^{-r}\log n(1-x^2)^{r/2}+\varepsilon_{n,r},$$

avec $|\varepsilon_{n,r}|<Crn^{-r}$, C: constante. D'autres résultats sont annoncés, concernant d'une part l'approximation en norme L, d'autre part le cas où $f^{(r)}$ est borné en norme $L^{(p)}$. *J. P. Kahane* (Montpellier).

Selivanova, S. G. Approximation by Fourier sums of functions having a derivative satisfying a Lipschitz condition. Dokl. Akad. Nauk SSSR (N.S.) 105 (1955), 909–912. (Russian)

Soit $W^{(r)}H^{(\alpha)}$ la classe des fonctions 2π-périodiques r fois dérivable, telles que $|f^{(r)}(x+h)-f^{(r)}(x)|<h^{\alpha}$, et soit $E_n=\sup|f-S_n|$ quand $f\in W^{(r)}H^{(\alpha)}$, S_n étant la $n$ième somme de Fourier de f. Précisant une estimation de Nikolsky [Trudy Mat. Inst. Steklov 15 (1945); MR 7, 435] l'auteur démontre une inégalité de la forme

$$|E_n-K(\alpha)n^{-r-\alpha}\log n|<Crn^{-r-\alpha},$$

C étant une constante absolue [pour le cas $\alpha=0$, une meilleure estimation est due à Sokolov, Dokl. Akad. Nauk SSSR (N.S.) 103 (1955), 23–26; MR 17, 361].

J. P. Kahane (Montpellier).

Walsh, J. L. Sur l'approximation par fonctions rationnelles et par fonctions holomorphes bornées. Ann. Mat. Pura Appl. (4) 39 (1955), 267–277.

Soit a_j $(j=1, \cdots, \mu)$ et b_k $(k=1, \cdots, \nu)$ des nombres complexes distincts, m_j et n_k des nombres >0, $\sum m_j=\sum n_k=1$; on pose $u(z)=\prod_{j,k}(z-a_j)^{m_j}(z-b_k)^{-n_k}$. Lemme fondamental: il existe une suite de fractions rationnelles $u_n(z)=P_n(z)/Q_n(z)$, où P_n (resp. Q_n) est un polynôme de degré n ne s'annulant pas hors des a_j (resp. b_k), telle que, sur tout compact ne contenant aucun a_j ni b_k, on ait $0<A_1<|u^{-n}(z)u_n(z)|<A_2$, A_1 et A_2 indépendants de n. Théorème 1 (cas particulier): soit E_ϱ le domaine $|u(z)|<\varrho$, et soit $f(z)$ holomorphe et bornée dans E_ϱ; alors $f(z)=\sum_0^{\infty}c_n u_n(z)$ dans E_ϱ, $|c_n|<A\varrho^{-n}$ et $|f(z)-\sum_0^n c_j u_j(z)|<A\sigma^n\varrho^{-n}$ dans E_σ quand $\sigma<\varrho$. Utilisant le th. 1 et son résultat sur la représentation conforme des aires multiplement connexes [C. R. Acad. Sci. Paris 239 (1954), 1572–1574; MR 16, 581 , l'auteur étudie (th. 2) l'approximation, par des fonctions holomorphes bornées, d'une fonction holomorphe donnée dans un domaine limité par des courbes de Jordan. *J. P. Kahane* (Montpellier).

Tureckiĭ, A. H. On an estimate of approximations by quadrature formulas for analytic functions. Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. 19 (1954), 32–40. (Russian)

The integral

$$U=\int_{|z-c|=r}\left|\int_a^b\frac{dx}{z-x}-\sum_{k=1}^{n}\frac{p_k}{z-x_k}\right|^2|dz|,$$

where $a\leqq x_k\leqq b$, $c=\frac{1}{2}(a+b)$, $r>\frac{1}{2}(b-a)$, may be used to obtain estimates of $\int_a^b f(x)dx-\sum_1^n p_k f(x_k)$ for functions $f(z)$ analytic in $|z-c|\leqq r$, in terms of $\int_{|z-c|=r}|f(z)|^2|dz|$. The problem of minimizing U by choice of the x_k, p_k is discussed and for fixed x_k, explicit formulae for minimizing p_k given. *G. G. Lorentz* (Detroit, Mich.).

Tureckiĭ, A. H. An example in the theory of approximation of functions. Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. 15 (1953), 26–31. (Russian)

The author discusses a method of constructing a function with a given degree of approximation by trigonometric polynomials of a given order, and with given Tchebyshev points. His argument is not satisfactory because of an incorrect characterization (on p. 28) of the polynomial of best approximation. *G. G. Lorentz.*

Gosselin, Richard P. On the convergence behaviour of trigonometric interpolating polynomials. Pacific J. Math. 5 (1955), 915–922.

Si $f(x)$ est une fonction continue, $I_n^{(\alpha)}(x; f)$ désigne le polynôme trigonométrique d'ordre n, prenant mêmes valeurs que $f(x)$ aux points $x=\alpha+2\pi j/(2n+1)$ $(j=0, \pm1, \cdots, \pm n)$. Marcinkiewicz [Acta Litt. Sci. Szeged 8 (1937), 131–135] a construit une fonction $f(x)$ telle que $I_n^{(0)}(x; f)$ diverge pour tout $x\neq 0$ (mod 2π). L'auteur construit une telle fonction avec, de plus, la propriété que $I_n^{(\alpha)}(x; f)$ converge uniformément pour un α donné incommensurable à π (resp. pour presque tout α).

J. P. Kahane (Montpellier).

Mikolás, Miklós. Über gewisse Eigenschaften orthogonaler Systeme der Klasse L^2 und die Eigenfunktionen Sturm-Liouvillescher Differentialgleichungen. Acta Math. Acad. Sci. Hungar. 6 (1955), 147–190. (Russian summary)

If one of the classical orthogonal systems is differentiated or integrated, after multiplication by a suitable function, there results again an orthogonal system, in general with a different weight-function; this property extends to Sturm-Liouville (SL) systems [see, e.g., Crum, Quart. J. Math. Oxford Ser. (2) 6 (1955), 121–127; MR 17, 266]. The author finds fairly exhaustive results on how far this situation is specific to the classical and to SL systems. Let either the $\{\phi_r(x)\}$, assumed non-constant, or γ_0, $\{\phi_r(x)\}$ $(r=1, 2, \cdots)$ be an orthonormal and complete system over a finite or infinite interval (a, b) with respect to a weight-function $w(x) \in L(a, b)$. Write $\Phi_r(x)=\int_a^x \phi_r(t)w(t)dt$. When is either $\{\Phi_r(x)+c_r\}$ or 1, $\{\Phi_r(x)+c_r\}$ an orthogonal system with respect to some $Q(x)$ (problem I), when is $\{\phi_r'(x)\}$ an orthogonal system with respect to some $g(x)$ (problem II), and when are both the case (problem III)? Problem I leads to an integral equation for the ϕ_r of the form $\phi_r(x)=\varrho_r\int_a^x Q(t)[\Phi_r(t)+c_r]dt+K_r$; this can be extended to the case of Lebesgue-Stieltjes weight-distributions, or specialised to a boundary problem for $(Q^{-1}\phi_r')'=\varrho_r w\phi_r$ in the case of suitable w and Q. Problem II leads to a boundary problem concerning $(g\phi_r')'=\varrho_r w\phi_r$ [see D. C. Lewis, Rend. Circ. Mat. Palermo (2) 2 (1953), 159–168; MR 15, 789]. Problem III leads to a special type of SL problem; the ϕ_r are to satisfy $(g\phi_r')'=\varrho_r w\phi_r$, and in addition

$$\lim_{x\to a+0} (\phi_m g\phi_n')=\lim_{x\to b-0}(\phi_m g\phi_n')$$

for $m, n=1, 2, \cdots$. Such a system is called "of ordinary SL-type", and the question of whether the systems $\{\phi_r'\}$, $\{\phi_r''\}$, $\cdots$ can simultaneously be of this type is studied. There follow a number of characterisations of classical systems, including those of Jacobi, Hermite and Laguerre. For example, the system $1, \cos x, \sin x, \cdots$ is determined, apart from order and apart from trivial linear transformations, as a complete orthogonal system over finite (a, b) of twice- differentiable functions $\phi_r\not\equiv 0$ with the initial conditions $\phi_r(a)\phi_r'(a)=0$, and such that the systems 1, $\{\phi_r'\}$, and 1, $\{\phi_r''\}$ are also orthogonal. Finally, there is a convergence theorem for the situation of problem II, with finite (a, b) and postulating also $\phi_r(a)=\phi_r(b)=0$; under certain additional assumptions, if $f(x) \in \mathrm{Lip}\, 1$ $(a\leq x\leq b)$, then $f(x)$ has a convergent expansion in the $\phi_r(x)$ whose formal derivative has the C_1-sum $f'(x)$ where $f'(x)$ exists and is continuous. This extends a theorem of L. Fejér [C. R. Acad. Sci. Paris 134 (1902), 762–765]. *F. V. Atkinson* (Canberra).

Popoviciu, Tiberiu. Sur la précision du calcul numérique dans l'interpolation par des polynomes. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 953–961. (Romanian. Russian and French summaries)

Let $x_1<x_2<\cdots<x_{n+1}$ be given and let $f_1, f_2, \cdots, f_{n+1}$ be corresponding given functional values. Denoting by D_i^j $(j=0, \cdots, n; i=1, \cdots, n+1-j)$ the divided difference of order j based on the points $x_i, \cdots, x_{i+j}$ $(D_i^0=f_i)$, let $L(x_0)=\sum_{j=0}^n (x_0-x_1)(x_0-x_2)\cdots(x_0-x_j)D_i^j$ be the value, at the given point x_0, of the interpolating polynomial $L(x)$, as given by Newton's formula. The problem is the following: Rounding off to a certain definite number of decimal places, let $f_i=\bar{f}_i+c_i^{(0)}$, where $\bar{f}_i$ is the truncated value and $c_i^{(0)}$ the error. The same is now done at every stage of the computation of the triangular scheme of divided differences: If $\tilde{D}_i^{j-1}$ are the truncated values of the differences of order $j-1$ as just obtained, let $\tilde{D}_i^j+c_i^{(j)}=(\tilde{D}_{i+1}^{j-1}-\tilde{D}_i^{j-1})/(x_{i+j}-x_i)$, where $\tilde{D}_i^j$ is the truncated value entered in the differencing scheme and $c_i^{(j)}$ is the rounding error. Thus the interpolated value as computed by Newton's formula is

$$\tilde{L}(x_0)=\sum (x_0-x_1)\cdots(x_0-x_j)\tilde{D}_1^j.$$

Let $L(x_0)=\tilde{L}(x_0)+\lambda$. Assuming that throughout the computation $|c_i^{(j)}|\leq\varepsilon$ (ε given, for instance $\varepsilon=10^{-k/2}$ if we work with k decimal places), what is an upper bound for the final error λ? This problem is elegantly solved as follows: Define a set of rational functions $N_k(x_1, x_2, \cdots, x_j)$ for $0\leq k<j\leq n+1$, by the recurrence relation

$$N_k(x_1, x_2, \cdots, x_{j+1})=$$
$$\{N_k(z_1, \cdots, x_j)+N_k(x_2, \cdots, x_{j+1})\}/(x_{j+1}-x_1)$$

if $k<j$, and $N_k(x, \cdots, x_{k+1})=1$. Then the above assumption $|c_i^{(j)}|\leq\varepsilon$ implies that $|\lambda|\leq V(x)\cdot\varepsilon$, where $V(x_0)=$

$$\sum_{i=0}^n |(x_0-x_1)(x_0-x_2)\cdots(x_0-x_i)| \sum_{r=0}^i N_{i-r}(x_1, x_2, \cdots, x_{i+1}).$$

A numerical example concludes the paper.

I. J. Schoenberg (Swarthmore, Pa.).

Šun, M. S. On a generalization of the Legendre polynomials. Har'kov. Gos. Univ. Uč. Zap. 29=Zap. Mat. Otd. Fiz.-Mat. Fak. i Har'kov. Mat. Obšč. (4) 21 (1949), 165–168. (Russian)

The polynomials discussed are the solutions of the recurrence relation $a_{n+1}X_{n+1}(z)-zX_n(z)+a_nX_{n-1}(z)=0$,

$$a_n=(n+\alpha)[(2n+2\alpha-1)(2n+2\alpha+1)]^{-\frac{1}{2}} \quad (n=1, 2, \cdots; 0\leq\alpha\leq 1),$$

generated by

$$X_0(z)=1 \text{ and } X_1(z)=[(2\alpha+1)(2\alpha+3)]^{\frac{1}{2}}(\alpha+1)^{-1}z.$$

$X_n(z)$ is a linear combination of $P_{n+\alpha}(z)Q_{\alpha-1}(z)$ and $Q_{n+\alpha}(z)P_{\alpha-1}(z)$. These polynomials form an orthonormal system on $[-1, 1]$ with a weight function $p_\alpha(z)$ which is explicitly given. When $\alpha=0$, they reduce to Legendre polynomials. *N. D. Kazarinoff* (Ann Arbor, Mich.).

See also: Fedulov, p. 1075; Tomaševič, p. 1059; Tureckiĭ, p. 1079.

Trigonometric Series and Integrals

Safronova, G. P. On determining the class of a trigonometric series. Leningrad. Gos. Univ. Uč. Zap. 144. Ser. Mat. Nauk 23 (1952), 102–110. (Russian)

Let B and C denote respectively the class of all bounded and continuous functions of period 2π. Familiar results give necessary and sufficient conditions for a given trigonometric series

$$(*) \qquad \tfrac{1}{2}a_0+\sum_1^\infty (a_n \cos nx+b_n \sin nx)$$

to be the Fourier series of a function of one of the classes B, C, L, L^p, or to be a Fourier-Stieltjes series; these results are expressed in terms of the $(C, 1)$ means $\sigma_n(x)$ of (*) [see e.g. the reviewer's "Trigonometrical series", Warsaw-Lvov, 1935, esp. Ch. IV, § 3]. The author proves

corresponding results by considering instead of the σ_n generalized linear means characterized by a matrix $\{\varrho_{mn}\}$ $(m, n=0, 1, \cdots)$ satisfying the conditions

$$\sum_n |\varrho_{mn}|<\infty \ (m=0,1, \cdots); \ \lim_{n\to\infty} \varrho_{mn}=1 \ (n=0, 1, \cdots),$$

$$\int_{-\pi}^{\pi} |\Sigma\, \tfrac{1}{2}\varrho_{m0}+\sum_{n=1}^{\infty} \varrho_{mn} \cos nx|dx<K \ (m=0, 1, \cdots).$$

A. Zygmund (Chicago, Ill.).

Kalašnikov, M. D. On some methods of approximation of continuous functions by trigonometric polynomials. Dopovidi Akad. Nauk Ukrain. RSR **1956**, 113–118. (Ukrainian. Russian summary)

Let $\tilde{C}$ be the class of all continuous functions of period 2π, and let $x_\nu=2\pi\nu/N$, where $\nu=1, 2, \cdots, N$ and $N\geqq 2n+1$. Let $T_n(x)=T_n{}^N(x, f)$ be the trigonometric polynomial of order $\leqq n$ which minimizes $\sum_{\nu=1}^N [f(x_\nu)-T_n(x_\nu)]^2$, and let $T_{n,m}{}^N$ be the mth partial sums of $T_n{}^N$. The author investigates the uniform convergence of the means

$\sigma_{n,p}{}^N=(p+1)^{-1}\{T_{n,n-p}{}^N+T_{n,n-p+1}{}^N+\cdots+T_{n,n}{}^N\}$,

$\tau_{n,p}{}^N=(p+1)^{-1}\{T_{n,0}{}^N+T_{n,1}{}^N+\cdots+T_{n,p}{}^N\}$,

where $p=0, 1, \cdots, n$ and $f\in\tilde{C}$, and obtains slight generalizations of results of S. N. Bernšteĭn [Dokl. Akad. Nauk SSSR **4** (1934), 1–8]. He also gives asymptotic expressions for the norms of the operator $\sigma_{n,p}{}^N$.

A. Zygmund (Chicago, Ill.).

Stečkin, S. B. On best approximation of conjugate functions by trigonometric polynomials. Izv. Akad. Nauk SSSR. Ser. Mat. **20** (1956), 197–206. (Russian)

Let $f(x)$ be a continuous function of period 2π, $\tilde{f}(x)$ its conjugate function, and $E_n(f)$ the best approximation to $f(x)$ by trigonometric polynomials of order n. The question of connecting the order of $E_n(\tilde{f})$ with that of $E_n(f)$ was studied by Bari [same Izv. Ser. Mat. **19** (1955), 285–302; MR **17**, 256]. The author takes up the same question again and carries it further by different methods. His principal result is that if r is a nonnegative integer and $\sum_1^\infty n^{r-1}E_n(f)<\infty$, then

$$E_n(\tilde{f}^{(r)})\leqq C_r\{(n+1)^r E_n(f)+\sum_{\nu=n+1}^{\infty} \nu^{r-1}E_\nu(f)\},$$

where C_r depends only on r. For $r=0$ this is shown to be essentially the best possible result. The author gives a number of corollaries. For example, if $\alpha>-1$ the series $\sum n^\alpha E_n(f)$ and $\sum n^\alpha E_n(\tilde{f})$ converge or diverge together. Another corollary is a theorem of Zygmund giving an estimate for the modulus of continuity of $\tilde{f}$ in terms of a majorant for the modulus of continuity of f [Prace Mat.-Fiz. 33 (1923–1924), 125–132].

R. P. Boas, Jr.

Satô, Masako. Uniform convergence of Fourier series. VI. Proc. Japan Acad. 32 (1956), 99–104.

The author improves his earlier theorems 3–6 [same Proc. 31 (1955), 600–605; MR **17**, 845] by replacing the condition $\int_0^{|h|} |f(x+u)-f(x)|du=o(|h|)$, as $h\to 0$, by

$$(*) \qquad \int_0^{|h|} (f(x+u)-f(x))du=o(|h|).$$

Thus his Theorem 6 becomes: If (*) holds at x and if

$$\int_0^{|h|} (f(t+u)-f(t-u))du=o(|h|/\log |h|^{-1}),$$

uniformly for all t, then the Fourier series of $f(t)$ converges at x.

Also, under condition (*), it is shown that

$$|s_n(x)|\leqq 16 \max_{\alpha\leqq x\leqq\beta}\left|(\beta-\alpha)^{-1}\int_\alpha^\beta f(t)dt\right|+O(1),$$

where $s_n(x)$ denotes the nth partial sum of the Fourier series of $f(t)$ at x.

W. W. Rogosinski.

Kovtun, D. G. On some series of the theory of heat conduction of Fourier-Poisson. I. Ukrain. Mat. Ž. **7** (1955), 273–290. (Russian)

The paper studies the convergence of the series solution, due to Poisson, of the equation of heat conduction with end conditions

$$\frac{\partial U}{\partial t}=a^2\frac{\partial^2 U}{\partial x^2}-bU, \ U_x(\pi, t)+HU(\pi,t)=0,$$

$$U_x(0, t)-hU(0, t)=0, \ U(x, 0)=f(x),$$

which has a solution of the form

$$U=e^{-bt}\sum e^{a^2\varrho_k{}^2 t}(A_k \cos \varrho_k+B_k \sin \varrho_k x),$$

where the ϱ_k are roots of a transcendental equation. By considering the Fourier transform in the complex plane of a function formed from f by continuation along the whole x-axis, and by careful consideration of the contour integrals the author shows that the expression for $U(x, t)$ is equal to a Weierstrass singular integral for f.

J. L. B. Cooper (Cardiff).

Kinukawa, Masakiti. Some strong summability of Fourier series. Proc. Japan Acad. 32 (1956), 86–89.

Let $s_n(x)$ be the nth partial sum of the Fourier series of $f(x)$ and let $\omega_p(t)=\sup_{|u|\leqq t}\{\int_0^{2\pi} |f(x+u)-f(x)|^p dx\}^{1/p}$. A partial extension is made of a theorem of Izumi [Tôhoku Math. J. (2) **5** (1954), 290–295; MR **16**, 240] by giving sufficient conditions for the almost everywhere convergence of $\sum |s_n(x)-f(x)|^k$ in terms of the convergence of sums of the type $\sum n^\alpha 2^{\beta n}\{\omega_p(2^{-n})\}^\gamma$.

P. Civin.

Izumi, Shin-ichi. Some trigonometrical series. XIX. Proc. Japan Acad. 32 (1956), 90–92.

Let $s_n(x)$ be the nth partial sum of the Fourier series of the almost everywhere differentiable function $f(x)$. If for $p>1$, $\beta>1$,

$$\left[\int_0^{2\pi} |f'(x+t)-f'(x-t)|^p dx\right]^{1/p}\leqq A[\log (1/t)]^{-\beta},$$

then the series $\sum |s_n(x)-f(x)|$ converges almost everywhere.

P. Civin (Eugene, Ore.).

Izumi, Shin-ichi. Some trigonometrical series. XX. Proc. Japan Acad. 32 (1956), 93–96.

Let $K>0$, and let $M=\{m_i\}$ be a collection of distinct integers, such that (*) the number of solutions of $m_i+m_j=m_k$, $i<j<k\leqq n$ is $o(n^2)$. Then there exists an integer $N(K)$ such that if $n\geqq N(K)$

$$\min_{0=x<2\pi} \sum_{i=1}^n \cos m_i x<-K.$$

This is a partial answer to a problem posed by Chowla [Bull. Amer. Math. Soc. **58** (1952), 287–305; MR **13**, 915] of whether the same result holds with no condition such as (*).

P. Civin (Eugene, Ore.).

Tureckiĭ, A. H. On an inequality of S. N. Bernšteĭn. Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. **15** (1953), 32–37. (Russian)

New proof of a theorem of S. Bernstein [Izv. Akad.

Nauk SSSR. Otd. Mat. Estest. Nauk (7) **1931,** 1151–1161] concerning the upper bound of the absolute value of a trigonometric polynomial of degree $(h-1)n$ satisfying $|T(k\pi/hn)|\leq 1$, $k=0, 1, \cdots, 2hn-1$. *G. G. Lorentz.*

Zygmund, A. On the Littlewood-Paley function $g^*(\theta)$. Proc. Nat. Acad. Sci. U.S.A. **42** (1956), 208–212.

Let $\Phi(z)=\sum_0^\infty c_n z^n$ be regular for $|z|<1$ and of class H^λ, $0<\lambda$, and $\Phi(e^{i\theta})=\lim_{r\to 1}\Phi(re^{i\theta})$. Let $\sigma_n^\alpha(\theta)$ be the nth (C, α) mean of the series $\sum_0^\infty c_n e^{in\theta}$ and let

$$M_a(f)=M_a(f(\theta))=\left\{(2\pi)^{-1}\int_0^{2\pi}|f(\theta)|^a d\theta\right\}^{1/a}.$$

Among the functions concerning which integral inequalities are obtained are the following:

$$g^*(\theta)=\left\{\int_0^1(1-\varrho)X^2(\varrho, \theta)d\varrho\right\}^{\frac{1}{2}},$$

where

$$X(\varrho, \theta)=\left\{\frac{1}{2\pi}\int_0^1\frac{(1-\varrho^2)|\Phi'(\varrho e^{i(\theta+t)})|^2}{(1-2\varrho\cos t+t^2)}dt\right\}^{\frac{1}{2}}.$$

Among the new inequalities obtained are

$$M\mu(g^*)\leq A\mu M_1(\Phi(e^{i\theta}))\ \ (0<\mu<1),$$

$$M_1(g^*)\leq A\int_0^{2\pi}\Phi(e^{i\theta})|\log^+|\Phi(e^{i\theta})|d\theta+B,$$

where in the last relation it is assumed that Φ belongs to $H\log^+ H$.

Also obtained are the inequalities

$$M_{\lambda,\eta}[\gamma]\leq C_{\lambda,\eta}M_\lambda(\Phi),\ \alpha=\lambda^{-1}-1\ \ (0<\lambda\leq 1),$$

$$M_\lambda[\gamma]\leq C_\lambda\left\{\int_0^{2\pi}|\Phi(e^{i\theta})|\ \log^+|\Phi(e^{i\theta})|d\theta+1\right\}^{1/\lambda}$$

for $\Phi\in H_\lambda\log^+ H$, which have also been observed by Sunouchi [Tôhoku Math. J. (2) **7** (1955), 96–109; MR **17,** 361]. These are applied to partially settle the conjecture of the author [Proc. London Math. Soc. (2) **47** (1942), 326–350; MR **4,** 76] that $\sup_r|\sigma_r^\alpha(\theta)|$ is in L^λ for $0<\lambda\leq\frac{1}{2}$ for Φ in $H^\lambda\log^+ H$. The case $\frac{1}{2}<\lambda<1$ remains open.
P. Civin (Eugene, Ore.).

Boas, R. P., Jr. Absolute convergence and integrability of trigonometric series. J. Rational Mech. Anal. **5** (1956), 621–632.

Izumi, Shin-ichi; and Tsuchikura, Tamotsu. Absolute convergence of Fourier expansions. Tôhoku Math. J. (2) **7** (1955), 243–251.

Independent studies are made in each of the above articles of the absolute convergence of Fourier cosine and sine expansions. Among the commonly obtained results are the following. Let $f(x)=\sum a_n\sin nx$ be of period π and suppose $\sum|a_n|<\infty$. Then the Fourier cosine expansion of $f(x)$ need not be absolutely convergent. It will be so if $\sum|a_n|\log n<\infty$ or $\sum n|\Delta a_n|<\infty$. If $f(x)=\sum b_n\cos nx$ is of period π with $f(0)=f(\pi)=0$ and $\sum|b_n|<\infty$, then the Fourier sine expansion of $f(x)$ need not be absolutely convergent. It will be so if $\sum|b_n|\log n<\infty$. The second article notes that the condition $\sum n|\Delta b_n|<\infty$ is not sufficient in this case.

The results in the first article are formulated in terms of $f(x)=\sum C_n e^{inx}$ with the series absolutely convergent. Let $f_+(x)=f(x)$, $0<x<\pi$ and $f_+(x)=0$, $-\pi<x<0$. If $f_+(x)$ has an absolutely convergent Fourier series then $x^{-1}f(x)$ is integrable. If $g(x)+\lambda x$ has an absolutely convergent Fourier series for some λ, then so does $f(x)g(x)$. If the even component $h(x)$ of $f(x)$ satisfies $h(0)=h(\pi)=0$, and if $\sum|C_m+C_{-m}|\log m<\infty$ then $f_+(x)$ has an absolutely convergent Fourier series. A partial converse is obtained for one of the quoted results in terms of lacunary series. Let $f(x)=\sum C(n_k)\sin n_k x$, $n_{k+1}/n_k\geq\theta>1$. Then, if both the Fourier sine and cosine expansions are absolutely convergent, then $\sum|C(n_k)|\log n_k<\infty$. *P. Civin.*

Satô, Masako. Lacunary Fourier series. II. Proc. Japan Acad. **31** (1955), 508–510.

Usant des résultats de la noté précédente [mêmes Proc. **31** (1955), 402–405; MR **17,** 478] l'auteur donne des conditions, portant sur les lacunes de la série de Fourier de $f(x)$ et sur le comportement de $f(x)$ au voisinage d'un point, pour que la série de Fourier de $f(x)$ soit absolument convergente. *J. P. Kahane* (Montpellier).

Boas, R. P., Jr. Isomorphism between H^p and L^p. Amer. J. Math. **77** (1955), 655–656.

Let L^p be the Banach space of complex-valued functions $f(x)$ of period 2π, with

$$\|f\|^p=\int_0^{2\pi}|f(x)|^p dx<+\infty,$$

and let H^p be the Banach space of functions $F(z)$, analytic in $|z|<1$, with

$$\|F\|^p=\sup_{0<r<1}\int_0^{2\pi}|F(re^{i\theta})|^p d\theta<+\infty.$$

The author shows that, if $p>1$, these two spaces are isomorphic in the sense of Banach (topologically isomorphic), i.e. there is a (1, 1) bicontinuous mapping of L^p into H^p.
F. Smithies (Cambridge, England).

Men'šov, D. On limits of sequences of partial sums of trigonometrical series. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 777–780. (Russian)

Let $S_n(x)$ be the partial sums of a trigonometric series (S) $\frac{1}{2}a_0+\sum_1^\infty(a_n\cos nx+b_n\sin nx)$. A measurable function $\varphi(x)$ defined on a measurable set $E\subset[-\pi, \pi]$ is said to be a limit function for S, if there is a sequence $n_1<n_2<\cdots$ such that $S_{n_k}(x)\to\varphi(x)$ almost everywhere on E (φ need not be finite almost everywhere on E). A family $\{\varphi\}$ of functions φ defined on E is said to be closed (with respect to convergence almost everywhere on E) if the family contains every function which is a limit (almost everywhere on E) of functions of the family. The present note contains statements, without proofs, about the families $\{\varphi\}$ of limit functions of a series S. The simplest, but not the most general result, of the note can be formulated as follows. A necessary and sufficient condition for a family $\{\varphi\}$ of measurable functions defined on a measurable set $E\subset[-\pi, \pi]$ to be the family of all limit functions for some series S is that $\{\varphi\}$ be closed.
A. Zygmund (Chicago, Ill.).

See also: Selivanova, p. 1077.

Integral Transforms, Operational Calculus

Puig Adam, P. On some properties of continued fractions of differential elements. Las Ciencias **20** (1955), 299–309. (Spanish)

The author surveys formal properties of his continuous continued fraction [given in earlier papers, see MR **13,** 540; **14,** 626; **16,** 804, 1019], and extends them to A. Stöhr's [Math. Nachr. **6** (1951), 103–107; **14,** 24] continuous continued fraction. *H. S. Wall* (Austin, Tex.).

Ivašev-Musatov, O. S. On Fourier-Stieltjes coefficients of singular functions. Izv. Akad. Nauk SSSR Ser. Mat. 20 (1956), 179–196. (Russian)

Let $F(x)$ be a non-decreasing, continuous, singular and non-constant function on $(0, 2\pi)$, and

$$c_n=\int_0^{2\pi} e^{-inx}dF(x)$$

its Fourier-Stieltjes coefficients. While, clearly, $\sum |c_n|^2=\infty$, it was known that given any function $r(n)$ increasing monotonically to $+\infty$ we can construct an F such that $c_n=O(r(n)n^{-\frac{1}{2}})$ (this result is due to Schaeffer [Amer. J. Math. **61** (1939), 934–940; MR **1**, 12] and is somewhat stronger than an earlier result of Wiener and Wintner [ibid. **60** (1938), 513–522]). The present note shows that the $|c_n|$ can tend to 0 as rapidly as

$$n^{-\frac{1}{2}},\ (n \log n)^{-\frac{1}{2}},\ (n \log n \log\log n)^{-\frac{1}{2}}, \text{ etc.}$$

More precisely, we have the following theorem, which slightly generalizes a result stated previously by the author without proof [Dokl. Akad. Nauk SSSR (N.S.) **82** (1952), 9–11; MR **14**, 163]. Suppose that a positive, monotonically decreasing and differentiable function $\chi(y)$ satisfies the conditions 1) $\int_1^y \chi^2(\eta)d\eta\to+\infty$ as $y\to+\infty$; 2) $y^{1+\varepsilon}\chi^2(y)\to+\infty$ as $y\to+\infty$, for each $\varepsilon>0$; 3) $y\chi^2(y)\to 0$ as $y\to\infty$; 4) there exists a y_0 such that for all $y>y_0$ and each $\theta\geqq 1$ we have $\chi(y)/\chi(\theta y)\leqq\theta^{\frac{1}{2}}$. Then there is a continuous non-decreasing and non-constant singular $F(x)$ such that

$$\int_0^{2\pi} e^{-inx}dF(x)=O(\chi(n))\}\quad (n\to+\infty).$$

A. Zygmund (Chicago, Ill.).

Avakumović, Vojislav G. Remark on Fatou-Riesz's theorem. Acad. Serbe Sci. Publ. Inst. Math. 8 (1955), 85–92.

The author's main result is clearly wrong as it stands but a change of $A(u)du$ to $dA(u)$ would make it plausible. Unfortunately the proof depends on an incorrect statement of a result of Paley-Wiener on the representation of a function as the Fourier transform of a function vanishing on a half line. *J. Blackman* (Ithaca, N.Y.).

Ahmad, Mansoor. On the expansion of a residual function as series of certain special forms. Ann. of Math. (2) 63 (1956), 549–564.

Let $P(e^{-z})$ be defined and differentiable for $-\infty<z<+\infty$, except possibly for a finite number of poles or isolated essential singularities, all inside the closed contour Δ; let $p(z)$ be the sum of the corresponding principal parts. If for some $\gamma>0$, $P(x)=O(x^\gamma)$, then

$$\chi_r(s)=\int_0^\infty e^{-sz}\{P(e^{-z})-p(z)\}dz+(2\pi i)^{-1}\int_\Delta e^{sz}p(z)dz$$

and

$$\chi_l(s)=\int_0^\infty e^{sz}\{P(e^{z})-p(-z)\}dz+(2\pi i)^{-1}\int_\Delta e^{-sz}p(-z)dz$$

each exist in certain halfplanes. If, furthermore, $\chi_r(s)=-\chi_l(s)$ in a certain common domain of existence, then $P(x)$ is called a residual function; the joint value of χ_r and χ_l is denoted by $\chi_0(s)$. Bochner [Ann. of Math. (2) **53** (1951), 332–363; J. Indian Math. Soc. (N.S.) **16** (1952), 99–102; MR **13**, 920; **14**, 967] first defined residual functions in connection with generalized "modular type" relations. If $s=\sigma+it$ and $\chi_0(s)$ satisfies also the additional conditions to be analytic and, for $-\infty<\sigma_1\leqq\sigma\leqq\sigma_2<\infty$, to be uniformly bounded as $t\to\infty$, then $P(x)$ is called residual in the ordinary sense. Using results of Bochner, the author finds necessary and sufficient conditions for a function to be residual in the general, or the ordinary sense. Some typical results are: If $P(e^{-z})$ is defined and differentiable for $-\infty<z<+\infty$, except, possibly, for a finite number of poles or essential singular points α_r $(r=1, 2, \cdots, n)$, of principal part $p_r((z-\alpha_r)^{-1})$, then a necessary and sufficient condition for $P(x)$ to be residual is that $P(x)=\sum_{r=1}^n p_r((\log x^{-1}-\alpha_r)^{-1})+P_1(x)$, where $P_1(x)$ is residual in the ordinary sense. If $P(e^{-z})$ is defined and differentiable in $-\infty<z<+\infty$, except for an essential singularity at the origin, then $P(x)$ is residual, if and only if it admits an expansion

$$P(x)=\sum_{n=1}^\infty \alpha_n(\log x^{-1})^{-n}+\sum_{n=0}^\infty \beta_n(\log x^{-1})^n/n!$$

with $\limsup_{n\to\infty} |\alpha_n|^{1/n}<\infty$, $\limsup_{n\to\infty} |\beta_n|^{1/n}<\infty$ so that

$$P(e^{-y})=f(y)=\sum_{n=1}^\infty \frac{\alpha_n}{y^n}\sum_{n=0}^\infty+\frac{\beta_n y^n}{n!}$$

and

$$\chi_0(s)=\sum_{n=0}^\infty \frac{\beta_n}{s^{n+1}}+\sum_{n=1}^\infty \frac{\alpha_n s^{n-1}}{(n-1)!}.$$

A necessary and sufficient condition for $P(x)$ to be residual in the ordinary sense is that it should admit an expansion of the form $P(x)=\sum_{n=0}^\infty c_n(\log x^{-1})^{-\frac{1}{2}}\cdot J_{n+\frac{1}{2}}(i\log x^{-1})$ in Bessel functions, $\gamma_0=\limsup_{n\to\infty} |c_n|/n<\infty$ and γ may be any number greater than $\frac{1}{2}\gamma_0+1$. Then

$$f(y)=P(e^{-y})=\sum_{n=0}^\infty c_n y^{-\frac{1}{2}}J_{n+\frac{1}{2}}(iy),$$

$$\chi_0(s)=\sum_{n=0}^\infty i^{n+\frac{1}{2}}2^{\frac{1}{2}}\pi^{-\frac{1}{2}}c_n Q_n(s),$$

where $Q_n(s)$ are the Legendre functions of second kind. *E. Grosswald* (Philadelphia, Pa.).

See also: Brachman and MacDonald, p. 1161; Fieber and Selig, p. 1092; Griffith, p. 1066; Stečkin, p. 1079; Thale, p. 1063.

Special Functions

Quilghini, Demore. Interpolazione di una funzione $F(P)$ continua nei punti P di una superficie sferica. Boll. Un. Mat. Ital. (3) 11 (1956), 40–45.

Let $0\leqq\theta\leqq\pi$, $0\leqq\phi\leqq 2\pi$ be spherical polar coordinates on the unit sphere. Let the zeros of the $(n+1)$st Legendre polynomial $P_{n+1}(\cos\theta)$ be $\theta_0, \theta_1, \cdots, \theta_n$. Put

$$h_\nu(\theta)=(1-2\cos\theta\cos\theta_\nu+\cos^2\theta_\nu)P_{n+1}{}^2(\cos\theta)$$
$$\times P'_{n+1}{}^{-2}(\cos\theta_\nu)\sin^{-2}\theta_\nu(\cos\theta-\cos\theta_\nu)^{-2}\ (\nu=0, 1, \cdots, n);$$
$$\phi_\mu=2\pi\mu/(m+1)\quad (\mu=0, 1, 2, \cdots, m);$$
$$k(\phi)=\left(\frac{\sin(\frac{1}{2}(m+1)\phi)}{(m+1)\sin(\frac{1}{2}\phi)}\right)^2,\ k_\mu(\phi)=k(\phi-\phi_\mu).$$

The author proves that for every continuous function $F(\theta, \phi)$ defined on the sphere

$$F(\theta, \phi)=\lim_{m,n\to+\infty}\sum_{\mu=0}^m\sum_{\nu=0}^n F(\theta_\nu, \phi_\mu)h_\nu(\theta)k_\mu(\phi)$$

except at the poles.

If $F(\theta, \phi)$ satisfies a Lipschitz condition

$$|F(\theta, \phi)-F(\theta', \phi')|\leq K\{|\cos\theta-\cos\theta'|^{\alpha}+|\phi-\phi'|^{\beta}\},$$

$\frac{1}{2}<\alpha\leq 1$, $0<\beta\leq 1$ the convergence is uniform on the whole sphere. *W. H. J. Fuchs* (Ithaca, N.Y.).

Łuszczki, Z.; Mikusiński, J.; Urbanik, K.; Wloka, J.; und Zieleźny, Z. Einige Bemerkungen über die Hirschman-Widder'schen Funktionen $H_{n,k}(x)$. Colloq. Math. 4 (1956), 30–32.

This is a brief study of the functions $H_{n,k}(x)$ introduced by the reviewer and D. V. Widder [Duke Math. J. 16 (1949), 433–438; MR 11, 29]. The definition given there is replaced by an equivalent definition from which the main properties can be deduced very easily. The same development. has been carried out independently by the reviewer and Widder [The convolution transform, Princeton, 1955; MR 17, 479]. *I. I. Hirschman.*

Lenz, H. Eine Bemerkung zur Einführung der Thetafunktionen. Jber. Deutsch. Math. Verein. 58 (1956), Abt. 2, 57.

This is a short proof of Jacobi's identity

$$\sum_{m=-\infty}^{\infty} q^{m^2}z^{2m}=\prod_{n=1}^{\infty}(1-q^{2n})F(z)\quad(|q|<1),$$

where $F(z)=\prod_{n=1}^{\infty}(1+q^{2n-1}z^2)(1+q^{2n-1}z^{-2})$. One shows that in

$$(*)\quad \sum_{m=-\infty}^{\infty} q^{m^2}z^{2m}=C\cdot F(z),\quad (**)\quad C=C(q)=\prod_{n=1}^{\infty}(1-q^{2n})$$

hold. The first equality (**) follows from $F(qz)=q^{-1}z^{-2}F(z)$. Setting in (*) successively $z=i$ and $z=e^{\frac{1}{4}\pi i}$, one obtains after simple manipulations that $C(q^{4^r})\prod_{n=1}^{\infty}(1-q^{4^r\cdot 2n})^{-1}$ is independent of the integral value r. As for $r\to\infty$ this constant is clearly one (use $|q|<1$), the last equality follows. *E. Grosswald* (Philadelphia, Pa.).

★**Jordan, Károly. Elliptikus függvények és alkalmazásuk. [Elliptic functions and applications.]** Tudományos Könyvkiadó, Budapest, 1950. 32 pp. 7.50 ft.

van der Pol, Balth. Démonstration élémentaire de al relation $\Theta_3{}^4=\Theta_0{}^4+\Theta_2{}^4$ entre les différentes fonctions de Jacobi. Enseignement Math. (2) 1 (1956), 258–261.

The set A of lattice points of the plane, of integral coordinates (m, n), can be divided into two subsets, according to the parity of $m+n$, the points in each set forming the lattices B_1 and B_2, respectively. If the function $f(m, n)$ is defined over A, then

$$\sum_A f(m, n)=\sum_{B_1} f(m,n)+\sum_{B_2} f(m, n),$$

provided that the last two sums converge separately; hence follows

$$\sum\sum f(m, n)=\sum\sum f(l+k, l-k)+\sum\sum f(l+k+1, l-k),$$

all limits of summation being $-\infty$ to $+\infty$. Applying this formula, in particular, to the Jacobi theta functions $\theta_i=\theta_i(0, \tau)$ $(i=0, 3, 2)$ and combining the results, one obtains $\theta_3{}^4=\theta_0{}^4+\theta_2{}^4$. This proof of the famous relation is completely elementary and does not use any of the properties of the elliptic functions. *E. Grosswald.*

San Juan, R. A special type of polynomials. Las Ciencias 19 (1954), 827–828. (Spanish)

The author gives a few properties of the polynomials $S_n(x)$ defined by

$$\left(x\frac{d}{dx}\right)^n e^{-x}=e^{-x}S_n(x).$$

A. Erdélyi (Pasadena, Calif.).

Denisyuk, I. M. Dynamical strains in the case of sudden loading at the lower end of a hoisting cable and polynomials analogous to those of Laguerre. Dopovidi Akad. Nauk Ukrain. RSR 1956, 127–129. (Ukrainian. Russian summary)

The author sets

$$m_n(x)=e^{-x}M_n(2x)=(-1)^n+\sum_{n=0}^{\infty} a_n\frac{(-t)^{n+1}}{n!(n+!)!},$$

where M_n is the polynomial defined in earlier papers [same Dopovidi 1954, 79–81, 165–167, 239–242; MR 16, 694], and he obtains a recurrence relation for the a_k as well as a representation in form of a determinant.

A. Erdélyi (Pasadena, Calif.).

Toscano, Letterio. Formule di riduzione tra funzioni e polinomi classici. Riv. Mat. Univ. Parma 6 (1955), 117–140.

The author considers the Hermite polynomials $H_n(x)$ which can be represented in terms of certain ${}_1F_1$ with first parameter $-n$, as well as the associated Hermite functions $h_n(x)$ representable in terms of certain ${}_1F_1$ with first parameter $-n+\frac{1}{2}$. It is shown that

$$h_\nu(x)H_{n+\nu}(x)-H_\nu(x)h_{n+\nu}(x)=\nu!e^{x^2/2}G_{n-1,\nu}(x),$$

where $G_{n-1,\nu}(x)$ is a certain polynomial of degree $n-1$. A recursion formula for this polynomial is found, integral formulas are established, and also further relations to the Hermite polynomials.

G. Szegö (Stanford, Calif.).

Danese, Arthur E. On a theorem of Merli concerning ultraspherical polynomials. Boll. Un. Mat. Ital. (3) 11 (1956), 38–39.

Improving a result of L. Merli [Atti 4° Congresso Un. Mat. Ital., Taormina, 1951, vol. 2, Edizioni Cremonese, Roma, 1953, pp. 151–155; MR 15, 121] the author proves the following. Let $P_n{}^{(\lambda)}(x)$ have the usual meaning, $F_n{}^{(\lambda)}(x)=P_n{}^{(\lambda)}(x)/P_n{}^{(\lambda)}(1)$,

$$\Delta_n{}^{(\lambda)}(x, k)=[F_n{}^{(\lambda)}(x)]^2-kF_{n-1}{}^{(\lambda)}(x)F_{n+1}{}^{(\lambda)}(x)\quad(\lambda>0, n\geq 1).$$

Then $\Delta_n{}^{(\lambda)}(x, k)\geq 0$ if $|x|\leq 1$, $0\leq k\leq 1$; moreover

$$\Delta_n{}^{(\lambda)}(x, k)<0$$

provided that $|x|>1$ and either

$$k<\frac{(n+2\lambda)(n+\lambda-1)}{(n+\lambda)(n+2\lambda-1)}\quad\text{or}\quad k\geq 1.$$

G. Szegö (Stanford, Calif.).

Petrašen', G. I.; Smirnova, N. S.; and Makarov, G. I. On asymptotic representations of cylinder functions. Leningrad. Gos. Univ. Uč. Zap. 170. Ser. Mat. Nauk 27 (1953), 7–95. (Russian)

The investigation consists of a minutely detailed application of the method of steepest descents to effect the appraisal of the Sommerfeld-Sonine integrals for various cylinder functions. In §§ 1 and 2 properties of these integrals are reviewed and derived. The method of

steepest descents is then described. The integrals

$$\pi i S_p^{(1)}(z)=\int_{(\lambda_0)} e^{-zf(w)}dw \text{ and } -\pi i S_p^{(2)}(z)=\int_{(\lambda_0^*)} e^{fz(w)}dw,$$

where $f(w)=w\cosh\gamma-\sinh w$, $\cosh\gamma=p/z$, and the integration paths are paths of steepest descent through w_0, $f'(w_0)=0$, are introduced. Asymptotic expansions for these integrals when $|p^2-z^2|>2.5|p|^{4/3}$ are then given and discussed. In §§ 3 and 4 Bessel functions of the first and third kinds of order p and argument z are identified with $S_p^{(1)}(z)$ and $S_p^{(2)}(z)$ by carefully deforming the contours in the Sommerfeld-Sonine integrals. The regions over which these identifications maintain are described. The cases $|\arg z|\leq\pi/4$ and $\pi/4<\arg z<3\pi/4$ are considered separately. In § 5 the results of the identifications made in §§ 3 and 4 are restated, and the curves of greatest increase of $S_p^{(1)}(z)$ and $S_p^{(2)}(z)$ in the ζ-plane, $\zeta=p/z$, are determined. Similar curves for $H_p^{(j)}(-ipz)$ and $J_p(-ipz)$, $j=1, 2$, are also derived.

The paper closes with five "applications". In the first two asymptotic forms for cylinder functions near $z=p$ are given up to terms of order $p^{-2/3}$. The remainder include four place tables of the real and imaginary parts of

$$\xi_0=-i[(1+z^2)^{\frac{1}{2}}-i \operatorname{arc} \cos i/z]$$

and

$$\Phi_0=(1-\zeta^2)^{\frac{1}{2}}-\zeta \operatorname{arc} \cos \zeta$$

and their derivatives to aid in locating zeros and computing values of $H_p^{(j)}(-ipz)$.

The paper was written by Petrašen'. The investigations of §§ 3 and 4 were completed independently by A. A. Zhdanov and G. I. Makarov on the one hand and N. S. Smirnova on the other. The applications 3-5 are due to the latter authors. The tables were constructed by Smirnova.

This research was partially motivated by the fact that earlier work as summarized by Watson [A treatise on the theory of Bessel functions, 2nd ed., Cambridge, 1944; MR 6, 64] is "insufficiently detailed and not completely correct." The authors do not specifically identify their improvements. The authors refer to the work of V. A. Fok [Tables of Airy functions, Inform. Otd. NII, Moscow, 1946] for asymptotic expansions of cylinder functions when $|p^2-z^2|>2.5|p|^{4/3}$. Such expansions are principal objectives in the work of Olver [Philos. Trans. Roy. Soc. London. Ser. A. **247** (1954), 328–368; MR **16**, 696] and Schöbe [Acta Math. **92** (1954), 265–307; MR **16**, 696] published after the paper under review. Olver gives expansions uniform in p and z for all the standard cylinder functions and their derivatives; these are applicable when $|p^2-z^2|$ is large or small. These papers also contain more complete work on zeros of cylinder functions.

The presentation often causes inconvenience. The results are set forth without references to the notation used in their statement. No distinction is made between the methods of steepest descents and stationary phase. A third terminology, "the method of crests", is often used. The numbering of equations is involved and confusing, and the definitions of a host of symbols lie buried in the text. Standard notations are often not used, e.g., for Airy functions. The same symbol is used with different meanings; e.g., $\zeta=p/z$ and $\zeta=\operatorname{Im} p$. The equality sign is repeatedly used when approximation is meant. At the top of p. 42 replace "<" by "≦". In the heading of § 4 replace "≦" by "<".

N. D. Kazarinoff.

Eweida, M. T. On an inequality concerning the derivatives of the Legendre polynomials. Rev. Mat. Hisp.-Amer. (4) **15** (1955), 161–164.

Let $f_{n,r}(x)$ be the rth derivative of the Legendre polynomial $P_n(x)$. Generalizing an inequality of Turán the author shows that

$$[f_{n,r}(x)]^2-f_{n-1,r}(x)\cdot f_{n+1,r}(x)>0 \quad (0\leq r\leq n-1),$$

for all $n\geq 1$ and $-1\leq x\leq 1$. *G. Szegö.*

Freud, G. Über differenzierte Folgen der Lagrangeschen Interpolation. Acta Math. Acad. Sci. Hungar. **6** (1955), 467–473. (Russian summary)

We assume that the weight function $w(x)$ satisfies the condition $w(x)\geq\mu>0$, $a\leq x\leq b$, where $\mu=\mu(a, b)$, $-1<a<b<1$. Let $f^{(m)}(x)$, $-1\leq x\leq 1$, be of the class Lip α, $\alpha>\frac{1}{2}$. We denote by $L_n(f; x)$ the Lagrange polynomials corresponding to the zeros of the orthogonal polynomials of $w(x)$. Then $\lim L_n^{(m)}(f; x)=f^{(m)}(x)$, uniformly in every (a, b). — A more general condition concerning $f^{(m)}(x)$ is also considered but then restrictions on $w(x)$ and the corresponding orthogonal polynomials must be imposed.

G. Szegö (Stanford, Calif.).

Carlitz, L. Note on a q-identity. Math. Scand. **3** (1955), 281–282 (1956).

Fjeldstad [Math. Scand. **2** (1954), 46–48; MR **16**, 3] has proved the formula

$$(*) \quad \sum (-1)^s\binom{m+n}{m+s}\binom{n+p}{n+s}\binom{p+m}{p+s}=\frac{(m+n+p)!}{m!n!p!},$$

where the summation is over all s yielding summands $\neq 0$. By specializing the parameters in a q-identity of Jackson [Quart. J. Math. Oxford Ser. **12** (1941), 167–172; MR **3**, 238] the author derives the following q-analog of(*):

$$\sum_{s=-m}^{m}(-1)^s\begin{bmatrix} m+n \\ m+s \end{bmatrix}\begin{bmatrix} n+p \\ n+s \end{bmatrix}\begin{bmatrix} p+m \\ p+s \end{bmatrix}q^{\frac{1}{2}(3s^2+s)} =\frac{[m+n+p]!}{[m]![n]![p]!}.$$

He also derives the q-analog of an identity equivalent to (*). *A. L. Whiteman* (Los Angeles, Calif.).

Have

★ **Erdélyi, A.; Kennedy, M.; and McGregor, J. L. Asymptotic forms of Coulomb wave functions. I. With an appendix by C. A. Swanson.** Tech. Rep. 4. Department of Mathematics, California Institute of Technology, Pasadena, 1955. 29 pp.

The Coulomb wave functions (CWF) are solutions of the differential equation

$$(*) \quad \frac{d^2y}{d\varrho^2}+\left[1-\frac{2\eta}{\varrho}-\frac{L(L+1)}{\varrho^2}\right]y=0 \quad (0<\varrho<\infty),$$

in which L and η are real parameters, $L\geq-\frac{1}{2}$, $\eta>0$. With $z=2i\varrho$, $k=i\eta$, and $m=L+\frac{1}{2}$, (*) reduces to the well-known Whittaker equation; the CWF are thus particular confluent hypergeometric functions. The authors give a complete description up to terms of order η^{-1} of the asymptotic behavior of the standard solutions of (*) for $\eta\to\infty$ and $\varrho\geq 2\eta$. The case $\varrho<2\eta$ is the subject of the next report in this series. Asymptotic forms of CWF have been obtained by many authors, but hitherto their behavior for large η and arbitrary positive ϱ has been incompletely understood. [For a review of the literature see H. Buchholz, Die konfluente hypergeometrische Funktion, Springer, Berlin, 1953; MR **14**, 978]. The con-

nection between the present results and related ones of Taylor [J. Math. Phys. **18** (1939), 34–49] is not explained.

The authors obtain their results by applying the theory of Langer [Trans. Amer. Math. Soc. **67** (1949), 461–490; MR **11**, 438] and Cherry [ibid. **68** (1950), 224–257; MR **11**, 596]. To make the application (*) is transformed into the equation

$$\frac{d^2y}{dx^2}+\nu^2\left[\frac{x}{1+x}-\frac{\alpha}{\nu^2(1+x)^2}\right]y=0,$$

wherein $2\eta=\nu$, $L(L+1)=\alpha$, $\varrho=\nu(1+x)$. The point $x=0$, where $\varrho=2\eta$, is a simple transition point of this equation. Thus away from $x=-1$ the asymptotic behavior of its solutions as $\nu\to\infty$ is expressible in terms of Airy functions. The dominant terms in the expansion for $F_L(\eta, \varrho)$ cancel to the left of the transition point giving rise to the need for other expansions when $\varrho<2\eta$.

It may be remarked that the methods of Langer and Cherry used by the authors do not require the reality of ϱ, η, and L; that is, the more general problem of the asymptotic behavior of Whittaker functions for large (complex) k and unrestricted (complex) z could be treated using these same methods.

The tables in the appendix give values of CWF for $L=0$, $\varrho=2\eta$, and $\eta=2(1)13$, $15(5)40$, $40(10)100$, $100(20)200$, and for $L=0$, $\eta=3$ or 5, and $\varrho=1(1)10$.

N. D. Kazarinoff (Ann Arbor, Mich.).

Hochstadt, Harry. Addition theorems for the functions of the paraboloid of revolution. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **BR-18** (1956), 1+22 pp.

Dans le but d'établir des formules d'addition pour les fonctions „du paraboloïde de révolution", l'auteur en reprend la définition à partir de l'équation des ondes $(\Delta^2+k^2)V=0$ qui se sépare dans les coordonnées paraboliques (ξ, η, φ) définies par:

(I) $\qquad x=2\sqrt{(\xi\eta)}\cos\varphi,\ y=2\sqrt{(\xi\eta)}\sin\varphi,\ z=\xi-\eta$

et admet des solutions qu'il est commode de prendre sous la forme:

$$\Omega_K^\mu(\xi, \eta, \varphi)=\Omega_K^\mu(P)=$$
$$\frac{\Gamma(\mu+n+1)}{n!}M^\mu{}_{n+\frac12(\mu+1)}(-2ik\xi)M^\mu{}_{n+\frac12(\mu+1)}(2ik\eta)e^{-i\mu\varphi},$$

$M_K{}^\mu$ désignant la fonction hypergéométrique confluente de Kummer. De la représentation intégrale connue de $M_K{}^\mu$:

$$M_K{}^\mu(z)=\Gamma\left(\tfrac12-K-\frac{\mu}{2}\right)e^{\frac12 z-\pi i(\frac12(1+\mu)+K)}I,$$
$$\left(I=\int_\infty^{(0+)}e^{-u}J_\mu(2\sqrt{(uz)})u^{K-\frac12(1+\mu)}du\right)$$

il déduit alors la fonction génératrice

$$G_\mu(x, y, z, t)=G_\mu(P, t)=\sum_0^\infty \Omega_n{}^\mu(P)(-t)^n$$

de $\Omega_n{}^\mu(P)$ qui s'écrit, en tenant compte de (I):

$$G_\mu(P, t)=t^{-\frac12\mu}\frac{e^{ikz}}{1+t}J_\mu\left(\frac{2k\varrho\sqrt{t}}{1+t}\right)e^{-i\mu\varphi}\quad (\varrho=\sqrt{(x^2+y^2)}).$$

Il n'est plus maintenant qu'à comparer les deux valeurs $G_\mu(P, t)$ et $G(_\mu P', t)$ pour en tirer des théorèmes d'addition sur les quantités $G_\mu(x, y, z, t)$, c'est-à-dire sur les quantités $M_K{}^\mu(\xi)$. C'est particulièrement intéressant, et assez simple, dans les cas où 1) P et P' sont sur Oz, 2) P et P' sont sur une même perpendiculaire à Oz, 3) P et P' sont sur un même cercle d'axe Oz.

R. Campbell (Caen).

See also: Ahmad, p. 1081; Carlitz, p. 1057; Inozemcev and Marčenko, p. 1076; Luke p. 1138; Smirnov, p. 1139; Šun, p. 1078; Vasil'ev, p. 1151; Hirschman, I., p. 1113.

Ordinary Differential Equations

Tizard, R. H. Note on initial conditions in the solution of linear differential equations with constant coefficients. Econometrica **24** (1956), 192–197.

Harlamov, P. V. On an estimate for the solutions of a system of differential equations. Ukrain. Mat. Ž. **7** (1955), 471–473. (Russian)

Consider two n-vector systems

$$\dot y=f(t, y)+\eta(t),\ \dot z=f(t, z)+\xi(t),$$

where $|\eta_i(t)|$, $|\xi_i(t)|<\varepsilon(t)$ with ε integrable. One supposes also that

$$|f_i(t, y)-f_i(t, z)|<L(t)\sum |y_i-z_i|,$$

where L is also integrable. If $\delta=\sup |y_i(t_0)-z_i(t_0)|$, the author proves

$$\|y(t)-z(t)\|\leq\left(n\delta+\left(\frac{2\varepsilon}{L}\right)e^{nL(t-t_0)}\right.$$

which improves a fundamental inequality in Nemyckiĭ and Stepanov, Qualitative theory of differential equations [Gostehizdat, Moscow-Leningrad, 1949; for a review of the 1st ed. see MR **10**,612

S. Lefschetz (Mexico, D.F.).

Sansone, Giovanni. Teorema di esistenza di soluzioni per un sistema di equazioni funzionali differenziali. Ann. Mat. Pura Appl. (4) **39** (1955), 65–67.

„Si dimostra col metodo di Tonelli, e con ipotesi più generali, un teorema di Joel Franklin [Proc. Amer. Math. Soc. **5** (1954), 363–369; MR **15**, 962] per il sistema di equazioni funzionali differenziali.

$$\frac{dy_\nu}{dx}=f_\nu[x; y_1(u_1(x)), \cdots, y_1(u_m(x)); \cdots$$
$$\cdots; y_n(u_1(x)), \cdots, y_n(u_m(x))]."$$

F. A. Ficken (Knoxville, Tenn.).

Samedova, S. A. Criteria of existence and uniqueness of a periodic solution of the equations $y'=f(x, y)$. Akad. Nauk Azerbaĭdžan. SSR. Trudy Inst. Fiz. Mat. **6** (1953), 25–39. (Russian. Azerbaijanian summary)

The equation $y'=f(x, y)$ is studied in the half-strip S: $x\geq0$, $a\leq y\leq b$. In most of the work it is assumed (conditions A) that $f(x, y)$ is continuous, is periodic in x, is monotonic in y, and has f_y bounded on S, and that the equation $f(x, y)=0$ has a unique continuous single-valued solution $y=\varphi(x)$, with $a\leq\varphi(x)\leq b$ for $x\geq0$. If $f_y<0(>0)$ then the curve $y=\varphi(x)$ is called a curve of stability (instability) and the region $\min\varphi(x)\leq y\leq \max\varphi(x)$, $x\geq0$, is called a region of stability (instability). If $f_y<0$, it is shown that as x increases any solution must enter the region of stability and no solution can leave it, that there exists a solution with the same period as that of f, and that if f_y is continuous and $f_y(x, y)<-F(x)$, where $\int^\infty F(x)dx=\infty$, then the periodic solution is unique

and every solution approaches the periodic one as $x\to\infty$. If $f_y>0$ then there is at least one bounded solution, and, if f_y is continuous and $f_y(x, y)>F(x)$, where $\int^\infty F(x)dx=\infty$, then there is exactly one bounded solution, and this solution is periodic. Further results are obtained in case $f(x, y)=0$ has several solutions, and the theory is applied to the Riccati equation.

F. A. Ficken (Knoxville, Tenn.).

Basov, V. P. On asymptotic behaviour of the solutions of systems of linear differential equations. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 951–954. (Russian)

The author gives a generalization of theorems proved in Prikl. Mat. Meh. **18** (1954), 313–328; Dokl. Akad. Nauk SSSR (N.S.) **80** (1951), 301–304; **81** (1951), 5–8 [MR **11**, 360; **13**, 557, 745]. The system of n first order linear differential equations is supposed to be already reduced to a somewhat extended L-diagonal form in the sense of I. M. Rapoport [On some asymptotic methods in the theory of differential equations, Izdat. Akad. Nauk Ukrain. SSR, Kiev, 1954; MR **17**, 734]. An m-dimensional subspace of vector solutions, $1\leqq m\leqq n$, is sought whose elements have limiting values as $t\to+\infty$. The elements $u(t)$ of the diagonal matrix (generalized characteristic root functions of t) are supposed to satisfy one of the conditions: (a) $\operatorname{Re}\int_{t_0}^t u(s)ds\to-\infty$ as $t\to+\infty$, or (b) $\operatorname{Re}\int_{t_0}^t u(s)ds>-c$ for some constant c.

L. Cesari.

Sobol', I. M. Limiting solution of Riccati's equation and its application to investigation of solutions of a linear differential equation of second order. Moskov. Gos. Univ. Uč. Zap. **155**, Mat. **5** (1952) 195–205. (Russian)

This paper consists primarily in supplying proofs for results previously reported by the author [Dokl. Akad. Nauk SSSR(N.S.) **65** (1949), 275–278; MR **10**, 536].

J. P. LaSalle (Notre Dame, Ind.).

Thomas, Johannes. Ein Abschätzungssatz für Lösungen Sturmscher Differentialgleichungen. Jber. Deutsch. Math. Verein. **58** (1956), Abt. 1, 110–114.

Let I be a closed interval of the real line on which $h(x)$ is continuous and the positive function $g_1(x)$ is twice continuously differentiable. Define $g(x)=(x-x_0)^\nu g_1(x)$, where $x_0\in I$ and ν is a positive even integer except when $x_0=\min_{x\in I} x$, in which case ν may be any positive number. The author determines an upper bound for $M(\lambda)=\max_{x\in I} u(x, \lambda)$ when λ is sufficiently large, where

$$d^2u/dx^2+[\lambda g(x)+h(x)]u=0,\quad u(a, \lambda)=0,\quad du(a,\lambda)/dx=1,$$

and $x, a\in I$. The estimate is made by comparing $u(x, \lambda)$ with a known solution of a related equation $d^2y/dx^2+[\lambda g(x)+h_0(x)]y=0$ according to the method of Langer [Trans. Amer. Math. Soc. **33** (1931), 23–64].

N. D. Kazarinoff.

Papuš, P. N. Study of the disposition of integral curves filling a region containing a singular point. Mat. Sb. N.S. **38(80)** (1956), 337–358. (Russian)

Let there be given an n-vector system of class C^2

$$\dot{x}=\varphi(x),$$

where the origin 0 is an isolated singularity. Let $V(x)$ of class C^2 be positive near 0 with $V(0)=0$ and suppose that

$$\Sigma\left(\frac{\partial V}{\partial x_i}\right)^2>0$$

everywhere except at the origin and that the manifolds $V=\varepsilon$ are topological spheres for ε small and positive [reviewer: the distance square r^2 is a suitable V]. Let $\dot{V}=F_1(x)$ (taken along the paths) and also $\ddot{V}=F_2(x)$. A ray is more or less a path $\to0$ as $t\to+\infty$ or $-\infty$. Elliptic solutions are solutions which for a certain T tend to the origin monotonely as $t\to-\infty$ or $+\infty$, for $t<T$ or $>T$. Hyperbolic solutions are defined in a fairly obvious way. The author assumes that the neighborhood of the origin is divided into a finite set of sectors in each of which F_1 has a fixed sign and gives a certain number of results regarding the phase portrait at the origin. Let $G_1, \cdots, G_s$ be the sectors and $C_1, \cdots, C_s$ their (local) boundaries on each of which one assumes that F_2 has a fixed sign (origin excepted). Theorem. If $F_2<0$ [>0] on all C_h-0 then the paths near 0 are either rays of else elliptic [hyperbolic]. Theorem. Consider the system

$$\dot{x}_i=\frac{\partial\Phi}{\partial x_i},$$

where $\Phi(x)$ is homogeneous of degree γ. Then for $\gamma<0$ [$\gamma>0$] all the paths near the origin are either rays or else elliptic [hyperbolic].

S. Lefschetz.

Al'muhamedov, M. I. Qualitative investigation of a system of differential equations. Kazan. Gos. Univ. Uč. Zap. **114** (1954), no. 8, 9–20. (Russian)

The system

$$(1)\qquad \dot{x}=\sum_{i=1}^n\frac{\lambda_i(x-x_i)}{(x-x_i)^2+(y-y_i)^2},\qquad \dot{y}=\sum_{i=1}^n\frac{\lambda_i(y-y_i)}{(x-x_i)^2+(y-y_i)^2}$$

describes the motion (filtration) of an underground layer of water or petroleum. The points $A_i(x_i, y_i)$ are the sources or sinks. For water the λ_i are <0, for petroleum they are >0. One may also write (1) as

$$\dot{x}=\frac{\partial P}{\partial x},\quad \dot{y}=\frac{\partial P}{\partial y},$$

$$P=\tfrac{1}{2}\log\prod\{(x-x_i)^2+(y-y_i)^2\}$$

or again by reducing the right hand sides to the same denominator as

$$\dot{x}=\frac{F(x, y)}{R(x, y)},\quad \dot{y}=\frac{G(x, y)}{R(x, y)}$$

with F, G, R polynomials. The paths are thus the same as for

$$\frac{dy}{dx}=\frac{G}{F}.$$

The author first studies the isobars $P=\text{const}$, then the paths. The A_i are nodes. There are other critical points which are saddle points. The infinite region is considered as a single point and it is a node. A few of the simpler systems are described explicitly.

S. Lefschetz.

Cambi, E. Possibility of free oscillations in a variable-parameter resonant system. Nuovo Cimento (10) 3 (1956), supplemento, 137–181.

This is a general discussion of linear physical systems, with one degree of freedom, such that one of the parameters, e.g. inductance, resistance or capacitance, is a given periodic function of the time. The theory of such a

system depends upon a linear differential equation, of the second order, with periodic coefficients. Substantially, the paper is a review of the known theory of such equations, with emphasis on the properties which are of importance in the physical applications. A relatively novel feature is an extensive discussion of the equation $(1+27\cos x)z''+p^2z=0$ paralleling the classical theory of the Mathieu equation $z''+p^2(1-2\gamma\cos x)z=0$. The paper concludes with a critical bibliography of the subject.

L. A. MacColl (New York, N.Y.).

Mitrinovitch, D. S. Compléments au Traité de Kamke. I. Jber. Deutsch. Math. Verein. **58** (1956), Abt. 2, 58–60.

L'auteur donne des exemples d'équations différentielles intégrables par la détermination de facteurs intégrands de la forme $F(xy)\times G(x^ry^s)$, $F(x^2y)G(xy^2)H(y/x)$.

R. Campbell (Caen).

Mitrinovitch, Dragoslav S. Compléments au traité de Kamke. II. Bull. Soc. Math. Phys. Serbie **7** (1955), 161–164 (1956). (Serbo-Croatian summary)

Cette courte note, partant d'une remarque de E. Kamke [Differentialgleichungen, Lösungsmethoden und Lösungen, Bd. I, Akademische Verlagsgesellschaft, Leipzig, 1942, pp. 478–480; pour une analyse de la 3ième éd. voir MR **9**, 33] considère l'équation:

$$\frac{d^2y}{dx^2}+\left[\frac{\alpha ff'}{\beta+f^2}-\frac{f''}{f'}\right]\frac{dy}{dx}+\frac{\gamma f'^2}{\beta+f^2}y=0$$

(où α, β, γ sont des constantes, f une fonction de x) et en étudie plusieurs cas d'intégrabilité par quadratures.

R. Campbell (Caen).

Semenov, N. Z. Three sufficient conditions for boundedness of the solutions of a linear equation of second order. Belorussk. Gos. Univ. Uč. Zap. Ser. Fiz.-Mat. **15** (1953), 38–40. (Russian)

The conditions given are not new, and simpler proofs have been given [see, e.g., R. Bellman, Stability theory of differential equations, McGraw-Hill, New York, 1953, Exercise 1, p. 138; MR **15**, 194]. *J. P. LaSalle.*

Shenitzer, Abe. Exponential solutions of second-order systems. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. **BR-17** (1956), i+11 pp.

Without giving attention to questions of convergence the solution of the $n\times n$-matrix differential equation (*) $dU/dt=A(t)U$, $U(0)=I$ can [cf. Magnus, Comm. Pure Appl. Math. **7** (1954), 649–673; MR **16**, 790] be written in the form $\exp\Omega(t)$, where $\Omega(t)=B_1+B_2+\cdots$ and $B_1=\int_0^t A(\tau)d\tau$, $B_2=\frac{1}{2}\int_0^t[(A(\tau),\int_0^\tau A(\tau)d\tau]d\sigma$, etc. Magnus has given sufficient conditions for almost all B_m to vanish, thus leaving for $\Omega(t)$ only a finite number of non-zero terms [cf. also Hellman, Div. Elektromag. Res., Inst. Math. Sci., New York Univ. Res. Rep. No. **BR-10** (1955); MR **16**, 670]. In the present paper it is shown that in the case of an analytic 2×2-matrix $A(t)$ either $\Omega(t)=B_1$ or Ω consists of infinitely many terms. Necessary and sufficient conditions are established for B_2 to vanish identically and $B_2\equiv0$ is found to be necessary and sufficient for all higher B_m to vanish. The method consists in working explicitly with the matrix elements and therefore might not be easily extendable to higher cases. However, in the last section of the paper a number of relations between the commutators of constant matrices are noted of which the following is an example: Let $P^{(0)}=P$, $P^{(k)}=[A_0,P^{(k-1)}]$ $(k=0,1,2,\cdots)$, then $[B,C]^{(k)}=\sum_{j=0}^k\binom{k}{j}[B^{(j)},C^{(k-j)}]$. These relations are applied to the coefficients of the matrix power series $A(t)=\sum_{m=0}^\infty A_mt^m$ and it is shown that $B_2=0$ implies $A_k^{(k-1)}=0$ $(k=2,3,\cdots)$ which had also been proved by Hellman (loc. cit.). The result is extended to higher "Lie-integral functionals" and thus a necessary condition is obtained for an $n\times n$-matrix differential equation (*) with analytic $A(t)$ to have an $\Omega(t)$ consisting of a finite number of terms only. *H. Schwerdtfeger.*

Galkin, M. S. On a solution of Cauchy's problem for a single equation. Prikl. Mat. Meh. **20** (1956), 271–278. (Russian)

The author considers the Cauchy problem for the fourth order equation (*) $(au'')''-bu=0$ on $[0,1]$ with $a>0$, $b>0$, a, b, a', b' continuous. It is shown that (*) has on $[0,1]$ a monotonically increasing solution and a monotonically decreasing solution, analogous to $e^{\pm x}$ for $u^{IV}-u=0$. The effect of small relative changes in initial conditions on these solutions is studied. Using the above monotonic solutions, the order of (*) is reduced to two, using Green's formula. Reduction of the order of a general linear equation using solutions of the adjoint is discussed.

W. S. Loud (Cambridge, Mass.).

Naĭšul', A. B.; and Svetlickiĭ, V. A. Determination of the configuration of the region of possible solutions of a system of linear differential equations. Prikl. Mat. Meh. **20** (1956), 144–147. (Russian)

The problem of determining a region of possible values at a given time T of the solutions to a linear system

$$(1)\qquad dx_i/dt=\sum_{j=1}^n A_{ij}(t)x_j+b_i(t)\quad(i=1,\cdots,n)$$

is discussed assuming the following: (a) the functions $A_{ij}(t)$ are known, (b) the region of possible initial values is known, and (c) the region in which the functions $b_j(t)$ lie is known and can be described in terms of linear combinations of functions whose upper and lower values at each time t are known. The principal matrix solution of the homogeneous system is used to express the components in a given direction of all the possible vector solutions of (1). Sufficient conditions for determining the maximum of these components are written down using Lagrange multipliers. The value of the paper is to make one realize that in attacking a problem of this type one need not restrict oneself merely to placing bounds on the components $x_i(t)$ and that information on the components in other directions can be used to further restrict the region of possible values. The authors fail to point out — what appears to be fairly obvious — that at best the method determines the convex hull of the region of all possible values. *J. P. LaSalle* (Notre Dame, Ind.).

Gambill, Robert A.; and Hale, Jack K. Subharmonic and ultraharmonic solutions for weakly non-linear systems. J. Rational Mech. Anal. **5** (1956), 353–394.

The authors consider systems of differential equations of the form

$$(*)\qquad \dot y_i=\varrho_iy_i+\varepsilon q_i(y_1,\cdots,y_n,t,\varepsilon)\quad(j=1,2,\cdots,n),$$

where ε is a small parameter, $\varrho_1,\cdots,\varrho_n$ are constants and each q_i is a complex valued function, periodic in the real variable t of period $2\pi/\omega$. The aim of the present paper is

to give general sufficient conditions for the existence of harmonic, subharmonic, ultraharmonic and ultra-subharmonic solutions of the system (*) for ε small. At the same time, the authors determine their approximations through a convergent process. In § 1, a method of successive approximations is defined and the convergence is proved. In § 2, general sufficient conditions are given for the periodic solutions of (*). In § 3, variations of the method of § 1 are discussed. In § 4, the authors give an existence theorem for real periodic solutions in the case where the classical Jacobian is not zero. § 5 contains sufficient conditions for real periodic solutions of the system of second order equations

$$\ddot{y}_i+\sigma_i^2 y_i=\varepsilon q_i(y_1, \cdots, y_n, \dot{y}_0, \cdots, \dot{y}_n, t, \varepsilon)\ (j=1, 2, \cdots, n)$$

for which the classical Jacobian may vanish. In § 6, Mathieu's nonlinear equation and Duffing's equation with large forcing terms are discussed. In § 7, Van der Pol's equation with a forcing term is discussed. In § 8, Reuter's equation is discussed. The theory may be applied to systems of differential equations. In this connection the authors discuss the case of two particular order systems in § 9. *M. Pinl* (Cologne).

Şaichin, A.; et Halanay, A. Sur l'équation de mouvement du train. Rev. Univ. "C. I. Parhon" Politehn. Bucureşti. Ser. Şti. Nat. 2 (1953), no. 3, 52–61. (Romanian. Russian and French summaries)

1. Let $du/ds=\Psi(u)+\Phi(s)$, assume Ψ, Φ continuous, $|\Phi(s)|\leqq M$, Ψ non-increasing in $[0, u_0]$, $\Psi(0)>M$, $\Psi(u_0)<-M$, and (H) there is a $\delta>0$ such that $\Delta\leqq\delta$ implies $\Psi(u+\Delta)<\Psi(u)$; then any two solutions differ by $<\delta$ when $s\to+\infty$, any solution satisfies $-\delta\leqq u\geqq u_0+\delta$ for sufficiently large s and there are supremum and infimum solutions contained in the strip $0\leqq u\leqq u_0$, $-\infty<s<+\infty$. 2. If instead of assumption (H) we suppose that $\Psi(u_2)-\Psi(u_1)<-k(u_2-u_1)^n$, $n\geqq 1$, $u_1\leqq u_2$, any two solutions are asymptotic to each other and there is a unique solution existing for all values of s. 3. If there is a solution bounded for $s\geqq s_0$, if $-K(u_2-u_1)<\Psi(u_2)-\Psi(u_1)<-k(u_2-u_1)$ when $m<u_1<u_2<M$, m, M being the bounds of the aforementioned solution, and if Φ is almost periodic, there is an almost periodic solution. *J. L. Massera.*

★ **Erugin, N. P. Methods for solving questions of stability in the large.** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 133–141. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)

A general lecture after the manner of the Symposium Lectures of the Amer. Math. Soc., on the work done recently in the USSR on the topic under consideration. There is given an extensive bibliography.

S. Lefschetz (Mexico, D.F.).

Krasovskiĭ, N. N. Inversion of theorems on Lyapunov's second method and questions of stability of motion in the first approximation. Prikl. Mat. Meh. 20 (1956), 255–265. (Russian)

This is a bird's eye view treatment of the stability properties at the origin of an n-vector system

$$\text{(1)}\qquad \dot{x}=X(x, t),\ X(0, t)=0,\ t\geqq 0,$$

in relation to the Lyapunov functions. There are first recalled the stability and asymptotic stability properties, uniform and otherwise (types A, B, C), then: (D). For δ suitably small and fixed and $\|x(t_0)\|<\delta$ we have

$$\|x(t)\|\leqq\|x(t_0)\|\cdot B\exp(-\alpha(t-t_0))$$

for all $t\geqq t_0$ (B, α are positive constants). There are then given two properties of trajectories: (C_1) given δ, $\eta>0$ there exists $T(\delta, \eta)\geqq 0$ such that for any t_0 and $\|x_0\|\leqq\eta$ at least one half trajectory through x_0 has points outside $\|x\|\leqq\delta$ for $|t-t_0|<T$. (D_1) For at least one half-trajectory $\|x\|\geqq\|x_0\|\cdot B\exp\alpha(t-t_0)$. Theorem 1. C_1 is a necessary and sufficient condition for the existence of a Lyapunov function of class C^1 in the neighborhood of the origin. Lemma. For autonomous or periodic systems C_1 is equivalent to: some neighborhood of the origin is free from complete trajectories other than the origin itself. Theorem 2. The origin is unstable if and only if there exists a function satisfying Lyapunov's second instability theorem. Theorem 3. A necessary and sufficient condition for uniform stability (relative to t_0) is the existence of a positive definite Lyapunov function $v\to 0$ uniformly in t for $t\to+\infty$ and with $\dot{v}<0$. Theorem 4. D_1 is equivalent to the existence of v such that

$$|v|<M\|x\|^A, <-k\|x\|^A; \left|\frac{\partial v}{\partial x}\right|<N\|x\|^{A-1}.$$

The paper deals also with extensions to retarded differential equations. There are 22 references. *S. Lefschetz.*

★ **Malkin, I. G. The present status of Poincaré's method and possibility of its utilization.** Trudy vtorogo vsesoyuznogo soveščaniya po teorii avtomatičeskogo regulirovaniya, Tom I [Transactions of the second all-union congress on the theory of automatic control, Vol. I], pp. 169–176. Izdat. Akad. Nauk SSSR, Moscow-Leningrad, 1955. (Russian)

A general lecture on the subject indicated in the title with emphasis on the application to automatic controls. Considerable information is given on research in this direction in the USSR (but nowhere else) in recent years. Excepting Poincaré there are 14 (Russian) references.

S. Lefschetz (Mexico, D.F.).

★ **Četaev, N. G. Ustoĭčivost' dviženiya. [Stability of motion.] 2d ed.** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 207 pp. 7 rubles.

This monograph is the second edition of one which appeared in 1946 and is well known in the Soviet Union but not outside. According to a statement by the author in the preface this edition does not differ very much from the first.

The monograph is an excellently written and reworked treatment of all the material in Lyapunov's great classic [Problème général de la stabilité du mouvement, Princeton 1947; MR 9, 34] which deals with his second method and readers familiar with Lyapunov will find no surprise here. There are a few new theorems due to Četaev himself and in particular his noteworthy general instability theorem. The point of view is generally physico-mathematical with emphasis on kinetic and potential energies as well as on the Lagrange equations. There are a number of well worked out examples, especially a highly interesting one on airplane stability. Of the extensive research on the inversion of the stability theorems of Lyapunov, all done since the first edition appeared, there is not a trace.

S. Lefschetz (Mexico, D.F.).

Kurzweil, Jaroslav. On the reversibility of the first theorem of Lyapunov concerning the stability of motion. Czechoslovak Math. J. 5(80) (1955), 382–398. (Russian. English summary)

The first stability theorem of Lyapunov may be defined as follows: Consider the n-vector system

$$\dot{x}=X(x, t),$$

where X is of class C^1 in $Q(H)$: $t\geqq 0$, $\sum x_i^2<H^2$, and $X(0; t)=0$ for $t\geqq 0$. If there exists in some $Q(h)$, $0<h\leqq H$ a function $V(x, t)$, such that $V(0, t)=0$, $t\geqq 0$, and which is positive definite, uniformly small and of class C^1 there, and if its time derivative $\dot{V}$ along the paths is $\leqq 0$ in $Q(h)$, then the origin is uniformly stable in the sense of Persidskiĭ. The inverse of this theorem is proved by the author. *S. Lefschetz* (Mexico, D.F.).

Kreĭn, M. G. On criteria of stable boundedness of solutions of periodic canonical systems. Prikl. Mat. Meh. 19 (1955), 641–680. (Russian)

The existence of a bounded solution of the vector system $dx/dt=JH(t)x$ is considered where x is a vector with $2m$ components. $H(t)$ is a real symmetric periodic matrix function of period T which is integrable over its period. J is the constant matrix $J=\begin{pmatrix}0 & I_m\\ -I_m & 0\end{pmatrix}$, where I_m is the unit matrix of m rows and columns. The matrix is said to be of positive type if for any vector $\zeta\neq 0$, $(H(t)\xi, \xi)\geqq 0$ and $\int_0^T (H(t)\xi, \xi)dt>0$. The above problem is studied by considering the boundary value problem

$$dx/dt=\lambda JH(t)x,\quad x(0)+x(T)=0.$$

Use is made of the relationship

$$\lim_{r\to\infty}\sum_{|\lambda_j|<r}\frac{1}{\lambda_j}=0,\quad \sum\frac{1}{\lambda_j^2}=\operatorname{tr}(A_{11}A_{22}-A_{12}^2),$$

where $\leqq\lambda_{-1}<0<\lambda_1\leqq\lambda_2\leqq\cdots$ are characteristic values and

$$\frac{1}{T}\int_0^T H(t)dt=\begin{pmatrix}A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}.$$

N. Levinson (Cambridge, Mass.).

Atrašenok, P. V. Some questions of the theory of stability of motion. Vestnik Leningrad. Univ. 9 (1954), no. 8, 79–106. (Russian)

This paper is concerned mainly with the system of differential equations

$$\frac{dx_i}{dt}=L_i(x)+\phi_i(x, t)+f_i(x, t),$$

where $x=(x_1, \cdots, x_n)$. The $L_i(x)$ are linear in x, for most applications with constant coefficients. It is assumed that there are constants $\varkappa$ and d such that

$$|\phi_i(x, t)|\leqq\varkappa|x|,\quad |f_i(x, t)|\leqq d.$$

N. Levinson (Cambridge, Mass.).

Erugin, N. P. On periodic solutions of differential equations. Prikl. Mat. Meh. 20 (1956), 148–152. (Russian)

Comments on a paper by the reviewer [Fac. Ingen. Montevideo Publ. Inst. Mat. Estadist. 2 (1950), 43–53; MR 13, 944]. Extensions to non-periodic systems are considered.

J. L. Massera (Montevideo).

Hale, Jack K. Periodic solutions of non-linear systems of differential equations. Riv. Mat. Univ. Parma 5 (1954), 281–311.

Let $\dot{x}=Ax+\varepsilon q(x, \varepsilon)$ be a system of differential equations, x, q being n-vectors, A matrix, q analytic, ε small parameter. The author describes a method of successive approximations to find the solutions near a given periodic solution of the equation for $\varepsilon=0$. Conditions for the existence of periodic solutions for $\varepsilon\neq 0$ are thus found.

J. L. Massera (Montevideo).

Halanay, A. Solution presque-périodiques pour une équation nonlinéaire contenant un petit paramètre. Rev. Univ. "C. I. Parhon" Politehn. București. Ser. Şti. Nat. 4 (1955), no. 6–7, 39–45. (Romanian. Russian and French summaries)

Let $\ddot{x}+p(t)x+\mu f(t, x, \mu)=0$ where p, f are almost periodic in t and f satisfies a Lipschitz condition with respect to x; assume that $\dot{y}+y^2+p(t)=0$ has two almost periodic solutions $u>0$ and $v<0$; then, if μ is small enough the given equation has one and only one almost periodic solution. A method of successive approximations for the calculation of this solution is described which converges geometrically if f satisfies certain more restrictive assumptions. *J. L. Massera* (Montevideo).

Spruch, Larry. On the eigenvalues which give upper and lower bounds on scattering phases. Div. Electromag. Res., Inst. Math. Sci., New York Univ., Res. Rep. No. CX-24 (1956), i+22 pp.

T. Kato [Progr. Theoret. Phys. 6 (1951), 394–407; MR 13, 655] has shown that upper and lower bounds on the scattering phases can be given by solving a certain associated eigenvalue problem. The conditions under which he could determine the eigenvalues are extended to include arbitrary angular momenta for non-negative potentials, certain cases for non-positive potentials, potentials solvable beyond $r=a$, and for less restricted potentials. *F. Rohrlich* (Iowa City, Ia.).

Mishoe, Luna I.; and Ford, Gloria C. On the limit of the coefficients of the eigenfunction series associated with a certain non-self-adjoint differential system. Proc. Amer. Math. Soc. 7 (1956), 260–266.

The differential system discussed is

$$\text{(I)}\qquad Au+\lambda By=0,\quad u(a)=u(b)=0,$$

where A denotes $d^2/dx^2+q(x)$, B denotes $-d/dx+p(x)$, and $q(x)$ and $p''(x)$ are continuous. The adjoint system is

$$\text{(II)}\qquad Av+\lambda B^*v=0,\quad v(a)=v(b)=0,$$

where B^* denotes $d/dx+p(x)$.

If the eigenfunctions of (I) and (II) are $u_n(x)$ and $v_n(x)$ respectively, the expansion of a function $F(x)$ of bounded variation in (a, b) is of the form $\sum_{-\infty}^{\infty}a_nu_n(x)$ where

$$a_n=\frac{\int_a^b F(\xi)B^*v_n(\xi)d\xi}{\int_a^b u_n(\xi)B^*v_n(\xi)d\xi}.$$

The series converges to $F(x)$ in (a, b) if $F'(x)$ exists and is of bounded variation in $a\leqq x\leqq b$ and

$$F(a)+F(b)\exp\left[-\int_a^b p(t)dt\right]=0.$$

But, unlike the ordinary Fourier coefficient, a_n does not necessarily tend to zero as n tends to infinity. It is shown here that the necessary and sufficient condition for a_n to tend to zero is that $F(a)=F(b)=0$. *E. T. Copson.*

Zatulovskaya, K. D. Clairaut systems of ordinary differential equations. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Nauki 5 (1954), 73–82. (Russian)

The classical results on the general and singular solutions of Clairaut's first-order implicit differential equation $y=xy'+\varphi(y')$ are generalized to the case where y is an n-dimensional vector. Two 2-dimensional examples are discussed. Though the study of this problem and the results are not new, no references are given. [See, e.g., A. C. Dixon, Philos. Trans. Royal Soc. London. Ser. A. **186** (1895), 523–565.]

J. P. LaSalle.

Feščenko, S. F. On the asymptotic decomposition of a system of linear differential equations. II. Estimate of error. Ukrain. Mat. Ž. 7 (1955), 443–452. (Russian)

The system under consideration (1) $dx/dt=a(\tau, \xi)x$ ($\tau=\xi^k t$, $1\leq k\leq n$, k an integer), and the stated decomposition of (1) into two systems (2) $d\zeta_\nu/dt=A_\nu(\tau, \xi)\zeta_\nu$, $\nu=1, 2$, of orders 2 and $n-2$, have been described in part I [same Ž. 7 (1955), 167–179; MR **17**, 365]. Both the matrices $A_\nu(\tau, \xi)$ of the systems (2) and the matrices $U_\nu(\tau, \xi)$, $\nu=1, 2$, by means of which the solutions x of (1) are obtained $x=U_1(\tau, \xi)\zeta_1+U_2(\tau, \xi)\zeta_2$, are given by means of formal series. By replacing A_ν, U_ν by means of the mth partial sums A_ν^m, U_ν^m of the corresponding series, and by supposing that the corresponding systems (2) are solved exactly, certain vectors x^m are obtained for which evaluations of the form $|x-x_m|\leq K_m\varepsilon^m$ are stated, where K_m is a constant independent from ε as $\varepsilon\to 0$.

L. Cesari.

Volosov, V. M. On some systems of differential equations having a small parameter. Dokl. Akad. Nauk SSSR (N.S.) **105** (1955), 397–400. (Russian)

In a previous paper [Mat. Sb. N.S. **31**(73) (1952), 645–674; MR **14**, 1086] the author studied the equation $\mu y^{(n)}+Q(x, y, \cdots, y^{(n-1)})=0$, where μ is a small parameter. In the present paper these results are generalized to systems of the form $\mu v'+Q(z, u_i, y_j, x)=0$, $z'=\alpha(x)v+\psi(z, u_i, y_j, x)$, $u_i'=\beta_i(x)v+\varphi_i(z, u_i, y_j, x)$, $y_j'=\lambda_j(z, u_i, y_j, x)$. The general appearance of the new theorems is very similar to that of the preceding ones but the complete statements fill up three whole pages and obviously cannot be summarized here. No proofs are given.

J. L. Massera (Montevideo).

Bureau, Florent. Le problème de Cauchy pour les systèmes d'équations linéaires aux dérivées partielles totalement hyperboliques. Ann. Mat. Pura Appl. (4) 39 (1955), 305–334.

This paper is a valuable summary, with an extensive bibliography, of a major portion of the author's work during the last decade. A historical introduction is followed by five chapters, the first of which is devoted to the finite (or logarithmic) part of certain divergent integrals. The second chapter discusses equations with constant coefficients. In the third and fourth chapters explicit expressions are developed for the solutions of two particular hyperbolic systems, each for two unknown functions with three and two space dimensions respectively, and with Cauchy data on the plane $t=0$ [cf. C. R. Acad. Sci. Paris **226** (1948) 1331–1333, MR **9**, (1948) 591]. In the brief fifth chapter the method of descent is applied to the particular problems just treated.

F. A. Ficken.

See also: Demaria, p. 1125; Fulton and Newton, p. 1166; Hu, Hai-Chang, p. 1155; Inaba, p. 1111; Puig, p. 1080; Zadiraka, p. 1138.

Partial Differential Equations

Pliś, A. The problem of non-local existence for solutions of a linear partial differential equation of the first order. Ann. Polon. Math. 2 (1955), 271–293 (1956).

The equation $Lz\equiv z_x+Q(x, y)z_y=0$ is studied on an open region D on which $Q \epsilon C'$. It was shown by Ważewski [Mathematica, Cluj 8 (1934), 103–116] that if $u \epsilon C'$ on all of D and $Lu=0$ then u is constant. It is shown here (Th. I) that if D is simply connected then there exists a u having a total differential on D, with $Lu=0$ and, almost everywhere on D, $u_y>0$. If D is finitely connected the theorem fails, but (Th. II) there still exists a u having a total differential on D, with $Lu=0$ and u constant on no open subset of D.

F. A. Ficken (Knoxville, Tenn.).

Pliś, A. On the uniqueness of the non-negative solution of the homogeneous Cauchy problem for a system of partial differential equations. Ann. Polon. Math. 2 (1955), 314–318 (1956).

A solution is sought for

$$\frac{\partial u_i}{\partial x}=\sum_{j=1}^{m}\sum_{k=1}^{n} a_{ijk}(x, y_1, \cdots, y_n)\frac{\partial u_j}{\partial y_k}+\sum_{j=1}^{m} b_{ij}(x, y_1, \cdots, y_n)u_j \qquad (i=1, \cdots, m)$$

subject to $u_i(0, y_1, \cdots, y_n)=0$ for $|y_k|\leq\beta$ ($k=1, \cdots, n$; $\beta>0$). Assuming that for $0<x\leq\alpha$ ($\alpha>0$) and $|y_k|\leq\beta$, the a's and b's are measurable with respect to $(y_1, \cdots, y_n)$, and are bounded, and that the a's satisfy Lipschitz conditions in each of the y's, it is shown that if u_i is of class C' in the pyramid R: $0<x\leq\gamma$, $|y_k|\leq\beta-cx$, (where γ and c are determined by other parameters in the problem) and continuous in the closure of the pyramid, if u_i solves the problem, and $u_i\geq 0$ in R, then $u_i\equiv 0$ in R. A theorem on nonlinear systems is also obtained.

F. A. Ficken (Knoxville, Tenn.).

Fujikawa, Hiroomi. The forces acting on two equal circular cylinders placed in a uniform stream at low values of Reynolds number. J. Phys. Soc. Japan **11** (1956), 558–569.

Rachajsky, B. Sur les systèmes en involution d'équations du second ordre. Bull. Soc. Math. Phys. Serbie 7 (1955), 11–20. (Serbo-Croatian. French summary)

Der Originaltitel dieser Arbeit ist allgemeiner gehalten als die hier angegebene Überschrift des Résumés und lautet: „Systeme partieller Differentialgleichungen zweiter Ordnung". Gleichwohl stehen unter diesen für Verfasser die bereits von S. Lie, G. Darboux, E. Goursat und N. Saltykow behandelten Involutionssysteme zweier partieller Differentialgleichungen zweiter Ordnung in zwei unabhängigen x, y und einer abhängigen Veränderlichen $z(x, y)$ im Vordergrunde, wie es die Überschrift des Résumés erwarten läßt. Ohne Einschränkung der Allgemeinheit kann man immer annehmen, daß zwei derartige Gleichungen nach $r=\partial^2z/\partial x_2$ und $t=\partial^2z/\partial y^2$ auf-

gelöst in der Gestalt:

$$(*)\quad r+f(x, y, z, p, q, s)=0,\ t+\varphi(x, y, z, p, q, s)=0,$$

$$p=\frac{\partial z}{\partial x},\quad q=\frac{dz}{\partial y},\quad s=\frac{\partial^2 z}{\partial x\,\partial y}$$

vorliegen. Ein vierparametrige Integralschar eines solchen Involutionssystems kann in der Gestalt

$$z=V(x, y, C_1, C_2, C_3, C_4)$$

angenommen werden. Dann liegt mit $\Delta_{xy}\neq 0$ für (*) und mit $\Delta_{x_1}\neq 0$ für das System

$$s=F(x, y, z, p, q, r)$$
$$t=\Phi(x, y, z, p, q, r)$$

ein vollständiges Integral vor, je nachdem ob der Ausdruck $\Delta_{x_1}\Delta_{y_1}-\Delta_{xy}^2$ identisch verschwindet oder nicht. Dabei bedeuten die Symbole Δ_{x_1}, Δ_{y_1}, Δ_{xy} die Funktionaldeterminanten

$$\Delta_{x_1}=\frac{\partial(V, V_x, V_y, V_{xx})}{\partial(C_1, C_2, C_3, C_4)},\quad \Delta_{y_1}=\frac{\partial(V, V_x, V_y, V_{yy})}{\partial(C_1, C_2, C_3, C_4)},$$

$$\Delta_{xy}=\frac{\partial(V, V_x, V_y, V_{xy})}{\partial(C_1, C_2, C_3, C_4)}.$$

(Die Elimination der Parameter C_1, C_2, C_3, C_4 führt auf die Gleichungen des Systems oder wenigstens auf Zweige dieser zurück). *M. Pinl* (Köln).

Moon, Parry; and Spencer, Domina Eberle. On the classification of the ordinary differential equations of field theory. Quart. Appl. Math. **14** (1956), 1–10.

The authors consider the problem of tabulation of those partial differential equations of mathematical physics which can be solved by separation of variables. The present study extends the general work of M. Bôcher [Über die Reihenentwicklungen der Potentialtheorie, Teubner, Leipzig, 1894] by actually finding the separation equations for the eleven simply-separable systems of Eisenhart, the eleven symmetric cyclide systems, and eighteen cylindrical systems. All separation equations for these coordinate systems reduce to nineteen distinct equations of the Bôcher type. Table 1 contains the classification of Bôcher equations that are obtained by separation of variables in 40 coordinate systems. Table 2 contains the separation equations of field theory.

M. Pinl (Cologne).

Smirnov, M. M. Functionally invariant solutions of an equation of hyperbolic-parabolic type. Leningrad. Inžen.-Èkonom. Inst. Trudy **1953**, no. 6, 239–244. (Russian)

Solution $u(x, y, t)$ of

$$(*)\quad L(u)=a_{11}\frac{\partial^2 u}{\partial x^2}+2a_{12}\frac{\partial^2 u}{\partial x\partial y}+a_{22}\frac{\partial^2 u}{\partial y^2}-\frac{\partial^2 u}{\partial t^2}+b_1\frac{\partial u}{\partial x}+ + b_2\frac{\partial u}{\partial y}-\beta\frac{\partial u}{\partial t}=0$$

(a_{11}, a_{12}, a_{22}, b_1, b_2, β are functions of x, y, t; $a_{11}>0$, $a_{12}>0$, $a_{22}>0$, and $a_{11}a_{22}-a_{12}^2=0$) is a functionally invariant solution if $F(u)$ is also a solution of (*) where F is an arbitrary twice differentiable function. A sufficient condition for the construction of a functionally invariant solution is derived by using the fact that a functionally invariant solution must satisfy the characteristic equation:

$$\left(\frac{\partial u}{\partial t}\right)^2-a_{11}\left(\frac{\partial u}{\partial x}\right)^2-2a_{12}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y}-a_{22}\left(\frac{\partial u}{\partial y}\right)^2=0;$$

and examples are given. For the case a_{11}, a_{12}, a_{22} constant and $b_1=b_2=\beta=0$, a formula for a solution satisfying initial conditions on u and $\partial u/\partial t$ is obtained.

J. Cronin (New York, N.Y.).

Brousse, Pierre. Quelques propriétés de la solution d'un problème singulier à un paramètre. C. R. Acad. Sci. Paris **242** (1956), 2093–2094.

Remarks on the first boundary-value problem for the equation

$$\Delta\Phi\ (x,y)-\frac{\lambda}{y^2}\Phi(x, y)=0\quad (\lambda\geqq -\tfrac{1}{4})$$

over a subregion of the upper half-plane. *M. G. Arsove.*

Magenes, Enrico. Sui problemi di derivata obliqua regolare per le equazioni lineari del secondo ordine di tipo ellittico. Ann. Mat. Pura Appl. (4) **40** (1955), 143–160.

L. Amerio [Amer. J. Math. **69** (1947), 447–489; MR **9**, 37] et G. Fichera [Ann. Mat. Pura Appl. (4) **27** (1948), 1–28; MR **10**, 534] ont étudié les problèmes de Dirichlet, de Neumann et de Dirichlet-Neumann pour un opérateur linéaire du second ordre elliptique, par une méthode due à M. Picone [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **2** (1947), 365–371, 485–492, 717–725; MR **9**, 145, 286, 287] consistant à ramener le problème à un système d'équations intégrales de Fischer-Riesz. L'auteur se place dans un ouvert borné de R^3 et développe parallèlement aux travaux cités la résolution, pour l'opérateur Δ du problème aux limites dans lequel on donne les valeurs de la fonction inconnue sur une partie de la frontière et celle de la dérivée oblique non tangentielle sur le reste de celle-ci, problème où surgissent des difficultés nouvelles par l'intervention d'intégrales principales et d'équations intégrales singulières. *H. G. Garnir.*

Hellwig, Günter. Anfangswertprobleme bei partiellen Differentialgleichungen mit Singularitäten. J. Rational Mech. Anal. **5** (1956), 395–418.

In dieser bemerkenswert inhaltsreichen Arbeit trifft Verfasser die wesentlichen Voraussetzungen über die von ihm behandelten Anfangswertprobleme: (1) der lineare partielle Differentialausdruck $\mathfrak{L}[u]$ ist für $x>0$ vom hyperbolischen Typus; (2) alle Abhängigkeitsgebiete sind endliche Gebiete. Als Anfangskurve $\mathfrak{k}$ sind die y-Achse oder geeignete Stücke dieser gewählt. Auf $\mathfrak{k}$ erfreuen sich die Koeffizienten von $\mathfrak{L}[u]$ beträchtlicher Freiheiten: $\mathfrak{L}[u]$ darf auf $\mathfrak{k}$ parabolisch werden, wodurch $\mathfrak{k}$ Träger der charakteristischen Spitzen oder Einhüllende der Charakteristiken wird. Dann handelt es sich um folgende Anfangswertprobleme:

$$(\mathrm{A}_1)\quad \lim_{x\to 0} u(x, y)=u_0(y),\ \lim_{x\to 0} u_x(x, y)=u_1(y),$$

$$(\mathrm{A}_2)\quad \lim_{x\to 0} u(x, y)=u_0(y),\ \lim_{x\to 0} u_x(x, y)=0,$$

$$(\mathrm{A}_3)\quad \lim_{x\to 0} u(x, y)=u_0(y),$$

$$(\mathrm{A}_4)\quad \lim_{x\to 0} u(x, y)=0,\ \lim_{x\to 0} u_x(x, y)=u_1(x, y),$$

$$(\mathrm{A}_5)\quad A_1 \text{ mit } f(u_0, u_1)=0.$$

Nur (A_1) ist das klassische Cauchysche Anfangswertproblem. (A_2) kommt in der klassischen Theorie nur bei

parabolischen Gleichungen vor. Das Ziel der Arbeit ist eine allgemeine Theorie dieser fünf Anfangswertprobleme. Die Koeffizienten A, B, C, P, Q, F des Differentialausdrucks $\mathfrak{L}[u]$ werden in der Form

$$A=x^a A_0(x,y),\ \cdots,\ F=x^f F_0(x,y)$$

(oder ähnlich) angenommen mit singularitätenfreien $A_0, \cdots, F_0$. Im Anschluß an bekannte Resultate von Diaz und Weinberger [Proc. Amer. Math. Soc. **4** (1953), 703–715; MR **15**, 321] und Weinstein [Proc. Symposia Appl. Math., v. 5, McGraw-Hill, New York, 1954, pp. 137–147; MR **16**, 137] wird nunmehr gezeigt, daß das Potenzverhalten der Koeffizienten allein im allgemeinen nicht genügt, um die Anfangswertprobleme zu klassifizieren. Ausnahmstypen dieser Art sind z.B.

$$\mathfrak{L}[u]\equiv u_{xx}-u_{yy}+\frac{\alpha(x,y)}{x}u_x+\beta(x,y)u_y+\gamma(x,y)u=f(x,y),$$

$$\mathfrak{L}[u]\equiv u_{xx}-u_{yy}x^2+\alpha(x,y)u_x+\beta(x,y)u_y+\gamma(x,y)u=f(x,y).$$

Unter geeigneten weiteren Voraussetzungen über die Koeffizienten wird deren Potenzverhalten für das Klassifikationsproblem charakteristisch und die Klassifikation von Verfasser vollständig durchgeführt. Die weitere Entwicklung der Theorie erfordert eine sorgfältige Untersuchung der jeweils vorliegenden charakteristischen Netze. Es folgt der Existenzbeweis für die sogenannten schwachen Lösungen der Anfangswertprobleme. Die schwachen Lösungen erweisen sich als Lösungen im üblichen Sinne, wenn man nachweist, daß auch die zweiten Ableitungen vorhanden sind. Insbesondere sind sie eindeutig bestimmt, hängen stetig von den Anfangsdaten ab und nehmen die Anfangsdaten im Sinne zweidimensionaler Konvergenz an. Als Beispiele werden behandelt:

$$(*)\quad \mathfrak{L}[u]= u_{xx}-u_{yy}+\frac{\alpha(x,y)}{x}u_x+\beta(x,y)u_y+\gamma(x,y)u=\mu(x,y),$$

$$(**)\quad \mathfrak{L}[u]=u_{xx}-x^c u_{yy}+(P_0(x,y)u)_x+ +(Q_0(x,y)u)_y-D_0(x,y)u=F_0(x,y).$$

(*) führt (unter jeweils geeigneten Stetigkeitsvoraussetzungen) auf (A_2) bzw. (A_1), (**) entsprechend auf (A_1). Für (A_5) können wegen der Kopplungsfunktionen keine allgemeinen Fallunterscheidungen gegeben werden.

M. Pinl (Köln).

Tomotika, S.; and Yosinobu, H. The flow of a viscous liquid past a flat plate at small Reynolds numbers. J. Math. Phys. **35** (1956), 1–12.

The authors repeat the earlier calculations of Piercy and Winny [Proc. Roy. Soc. London. Ser. A. **140** (1933), 543–561], Davies [Phil. Mag. (7) **31** (1941), 283–313; MR **2**, 328] and Sidrak [Proc. Roy. Irish Acad. Sect. A. **53** (1950), 65–81; MR **12**, 137]. The results on drag of the earlier works were known to be different from each other. The careful analysis of the present authors shows that the skin-friction coefficient up to the second approximation agrees perfectly with that of Piercy and Winny.

Y. H. Kuo (Ithaca, N.Y.).

Kawaguti, Mitutosi. On the viscous shear flow around a circular cylinder. II. Oseen's approximation. J. Phys. Soc. Japan **11** (1956), 570–583.

[For part I see Rep. Inst. Sci. Tech. Univ. Tokyo **6** (1952), 85–91; MR **14**, 104.] The author considers a viscous flow around a circular cylinder at small Reynolds number with a simple shear flow at infinity. The calculation for the effects of the simple shear was carried out to the first order of a parameter, characterizing the vorticity in the simple shear. *Y. H. Kuo* (Ithaca, N.Y.).

Weinstein, Alexander. The generalized radiation problem and the Euler-Poisson-Darboux equation. Summa Brasil. Math. **3** (1955), 125–147.

The author gives a new formulation to the radiation problem so that all problems treated in his paper appear as problems for the Euler-Poisson-Darboux equation

$$(*)\qquad u_{xx}-u_{yy}-\frac{k}{y}u_y=0$$

in two independent variables. The unified solution appears in a different form from the special cases given by R. Courant and D. Hilbert [Methoden der mathematischen Physik, vol. II, Springer, Berlin, 1937] or P. Germain and R. Bader [Rend. Circ. Mat. Palermo (2) **2** (1953), 53–70; O.N.E.R.A. no. **54** (1952); MR **15**, 876; **14**, 654]. The method, nevertheless, is equivalent to the solution of Germain and Bader and yields a correction of the formula as given in Courant Hilbert Vol. II.

The new formulation of the radiation problem is to find a twice differentiable solution of the Euler-Poisson-Darboux equation (*) which satisfies the following boundary conditions: $u(x,0)=f(x)$, $u(x,y)=0$ on the characteristic $y=x$. The generalized radiation problem is, for the value $k=\frac{1}{3}$, identical with a problem for Tricomi's equation

$$z\omega_{xx}+\omega_{zz}=0,\quad \omega=\omega(x,z).$$

The Radiation-Tricomi-Germain-problem (RTG-problem) can be solved by the recursion formula $u_y^{(k)}(x,y)=yu_{(x,y)}^{(k+2)}$ [cf. A. Weinstein, C. R. Acad. Sci. Paris **234** (1952), 2584–2585; Comm. Pure Appl. Math. **7** (1954), 105–116; MR **14**, 176; **16**, 137].

The author gets final results for $k=-2k$ and for all values $k<1$. The formula for the RTG-Problem given by Germain and Bader for $k=\frac{1}{3}$ can be extended to the value $0<k<1$. In a subsequent paper the uniqueness of the solution of the RTG-problem will be investigated.

M. Pinl (Cologne).

Zeragiya, P. K. Solution of fundamental boundary problems for non-linear differential equations of parabolic type by the method of academician S. A. Čaplygin. Soobšč. Akad. Nauk Gruzin. SSR **17** (1956), 103–109. (Russian)

The author applies under and upper functions to establish the existence up to an arbitrary time T of a solution of $\Delta u-\partial u/\partial t=f(x,t,u)$, where x is an n-dimensional vector, u vanishes initially, and either u or its normal derivative or a linear combination of both vanishes on the boundary of the region. The function f is assumed to satisfy a Lipschitz condition on the closed region and $f_u>0$ for finite u. *A. S. Householder.*

Allan, Douglas. The solution of a special heat and diffusion equation. Amer. Math. Monthly **63** (1956), 315–323.

The boundary value problem $\rho c\theta_t=K(\theta_{rr}+2r^{-1}\theta_r)+F(r,t)$, $\theta(a,t)=\theta(r,0)=0$, $\theta(0,t)$ bounded, where $F=0$ when $0<r<r_1$, $F=H_0e^{-\lambda t}$ when $r_1<r<a$, is solved by two methods. An application of the result to the diffusion of a gas being produced within a sphere by radioactive

decay, an application pertaining to age-determination of mineral samples, is discussed briefly. *R. V. Churchill.*

Fieber, H.; und Selig, F. Temperaturfelder in endlichen Körpern bei bewegten Wärmequellen. Österreich. Ing.-Arch. **10** (1956), 96–103.

Four formulas are derived for transient temperature distributions in bounded media heated by point or line sources that move with constant velocity. The simplest formula is one for the temperatures $T(\phi, t)$ in a thin circular ring $0 \leq \phi < 2\pi$ heated by a point source that moves with constant angular velocity ω around the ring, when surface heat transfer takes place into a medium at temperature zero. Thus $T_t = a^2 T_{\phi\phi} + \alpha_1 \delta(\phi - \omega t) - \alpha_2 T$, $T(\phi, 0) = 0$, where the α's denote constants and δ is the unit impulse function. The formal application of the finite exponential Fourier transformation with respect to ϕ leads directly to a series representation for the function $T(\phi, t)$. A special case of that function is examined graphically. Similarly, a formula for the temperatures $T(x, y, z, t)$ in a medium $|x| < L_1$, $|y| < L_2$, $|z| < L_3$ is derived when a line source moves across the mid-section $x = 0$, so that $T_t = a^2 \nabla^2 T + \alpha_1 \delta(x) \delta(y + L_2 - vt)$, where v is a constant. Surface heat transfer takes place at the six faces. Modified finite Fourier transformations are applied with respect to each of the variables x, y and z. The remaining two formulas give the temperatures in hollow circular cylinders of finite length. In the first case a point source moves along an element of the outer cylindrical surface. In the second case, the source moves around the outer boundary of the middle section. Here appropriate finite Hankel transforms are used. All four formulas represent the temperatures by means of infinite series.

R. V. Churchill (Ann Arbor, Mich.).

Fourès-Bruhat, Yvonne. Solution élémentaire d'équations ultrahyperboliques à coefficients variables. C. R. Acad. Sci. Paris **242** (1956), 1566–1568.

Recherche d'une solution élémentaire de l'opérateur $L=$

$$\sum_{i,j=1}^{n_1} A^{ij} \frac{\partial^2}{\partial x_i \partial x_j} - \sum_{\alpha,\beta=1}^{n_2} A^{\alpha\beta} \frac{\partial^2}{\partial x^\alpha \partial x^\beta} + \sum_{r=1}^{n-n_1+n_2} B^r \frac{\partial}{\partial x_r} + C, x \in R^n,$$

dont les coefficients sont suffisamment réguliers, les formes quadratiques $\sum A^{ij} x_i x_j$ et $\sum A^{\alpha\beta} x_\alpha x_\beta$ étant définies positives. En transformant l'opérateur étudié dans des variables liées à l'équation du cône caractéristique Γ_M de l'opérateur L, issu du point M, l'auteur obtient d'abord une identité vérifiée par la restriction à Γ_M des fonctions définies dans R^n. Les coefficients de cette relation résultent de la résolution d'un système récurrent d'équations différentielles. De cette première identité, il en déduit une autre vérifiée par les solutions de l'équation $Lu = \psi$, fonction indéfiniment dérivable à support compact. Cette dernière identité, considérée comme une équation intégrale, détermine la solution élémentaire cherchée.

H. G. Garnir (Liège).

Ovsyannikov, L. V. On linearization of a second-order partial differential equation. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 219–221. (Russian)

Let u be a function of n independent variables x^i and consider the partial differential equation

$$F(u_{ij}, u_i, u, x^i) = 0 \tag{1}$$

in which u is replaced by a new function v through the substitution

$$v = N(x^i; u). \tag{2}$$

Equation (1) becomes

$$L(v) = a^{ij} v_{ij} + 2b^i v_i + kv + 1 = 0 \quad (a^{ij} = a^{ji}) \tag{3}$$

only when (1) is of the form

$$a^{ij} u_{ij} + 2B^i u_i + C a^{ij} u_i u_j + D = 0, \tag{4}$$

where the a^{ij} depend only on the x^i and B^i, C, D on x^i and u. (Summation from 1 through n is indicated by repeated indices.)

The main results of this paper are summarized in the following theorem. In order that there exist a substitution of the form (2) transforming (4) to a linear equation in v, it is necessary and sufficient that

$$a^{ij} C_i = B_u{}^i \quad (i = 1, 2, \cdots, n),$$
$$a^{ij} C_{ij} + 2B^i C_i = (D_u + CD)_u.$$

If these conditions are satisfied, then the desired substitution (2) is given by any nontrivial solution of

$$N_{uu} = C N_u.$$

C. G. Maple (Ames, Ia.).

Èĭdel'man, S. D. On analyticity of solutions of parabolic systems. Dokl. Akad. Nauk SSSR (N.S.) **103** (1955), 27–30. (Russian)

Regular solutions of the homogeneous parabolic system of equations $\partial u/\partial t = A(t, x, \partial/\partial x)u$ must be analytic in the space variables when A is a strongly elliptic matrix of differential operators of order $2b$ having coefficients depending smoothly on t and analytically on the space variables $x = (x_1, \cdots, x_n)$. A regular solution is one admitting enough derivatives to make both sides of the equation meaningful, and with a growth $|u(x, t)| \leq M \exp(K|x|^q)$, $q = 2b(ab-1)^{-1}$, M and K constants. The author proves this theorem following methods of I. Petrofsky who treated the case where $b = 1$ and the coefficients depend only on t. New estimates for his fundamental solution [same Dokl. (N.S.) **98** (1954), 913–915; MR **17**, 857] are derived, by applying the theorem of Phragmen-Lindelöf to the fundamental solution for systems with coefficients depending only on t, and these in turn applied to the integral equations defining the general fundamental matrix solution. The case of inhomogeneous systems is also discussed, as well as certain more general systems involving derivatives with respect to t of order higher than the first.

A. N. Milgram (Minneapolis, Minn.).

Borovikov, V. A. Generalization of the Herglotz-Petrovskiĭ formula and the diffusion of waves. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 587–590. (Russian)

On considère l'opérateur différentiel $\Delta(\partial/\partial t, \partial/\partial x_i)$, $i = 1, \cdots, p$, homogène de degré n, à coefficients constants, l'équation $\Delta(\tau, \xi_i) = 0$ admettant n racines en τ réelles et distincts. On suppose que la surface $\Delta(1, \xi_i) = 0$, ξ_i réel, est bornée. L'A. étudie la diffusion des ondes [cf. Hadamard, Le problème de Cauchy⋯, Hermann, Paris, 1932], i.e. le support de la solution élémentaire $K(x, t)$. A l'aide de variantes des expressions de $K(x, t)$ données par Herglotz [Ber. Verh. Sächs. Akad. Wiss. Leipzig. Math.-Phys. Kl. **78** (1926), 93–126, 287–318; **80** (1928), 69–114; Petrowsky, Mat. Sb. N.S. **17(58)** (1945), 289–370; MR **8**, 29; Gel'fand et Šapiro, Uspehi Mat. Nauk (N.S.) **10** (1955), no. 3(65), 3–70; MR **17**, 371; cf. aussi Leray,

Hyperbolic differential equations, Inst. Advanced Study, Princeton, 1953; MR 16, 139] l'A. démontre que, sous les hypothèses ci-dessus, la condition nécessaire et suffisante pour qu'il y ait absence de diffusion des ondes est que p soit impair et $n-p-1<0$. *J. L. Lions* (Nancy).

Leray, Jean. Le problème de Cauchy pour une équation linéaire à coefficients polynomiaux. C. R. Acad. Sci. Paris **242** (1956), 953–959.

L'auteur étudie un problème de Cauchy local relatif aux opérateurs aux dérivées partielles linéaires à coefficients polynomiaux, problème consistant à trouver $u(x)$ tel que $a(x, \partial/\partial x)u(x)=v(x)$ (a: polynôme en x et $\partial/\partial x$), la fonction $v(x)$ étant holomorphe au voisinage d'un morceau d'hypersurface non caractéristique de l'espace affine X à l dimensions sur le corps des nombres complexes. Si $s(x)$ est une fonction holomorphe qui s'annule sur l'hypersurface considérée, on impose à $u(x)$ de s'annuler avec $s(x)$ un nombre de fois égal à l'ordre de a. L'auteur résout ce problème par une opération fonctionnelle, définie par une quadrature, qui ne dépend que de l'hypersurface $s(x)=0$. Cette transformation opère sur la solution d'un problème de Cauchy particulier relatif à un autre opérateur linéaire aux dérivées partielles à coefficients polynomiaux, l'ordre de ce dernier étant le degré de a et inversément, les données étant prises sur un hyperplan arbitraire.

L'auteur étudie en particulier les équations à coefficients linéaires. Dans ce cas, la résolution du problème de Cauchy exige seulement une quadrature et l'intégration d'un système différentiel. Si les coefficients sont constants, il obtient une formule résolutive où n'intervient qu'une quadrature, déjà connue de L. Fantappiè sous une autre forme [Ann. Mat. Pura Appl. (4) **22** (1943), 181–289; MR **8**, 589]. Enfin, l'auteur explicite les rapports existant entre le produit fonctionnel projectif de L. Fantappiè et l'opération fonctionnelle utilisée dans la présente note.

H. G. Garnir (Liège).

Perčinkova-V'čkova, Danica. Sur deux équations aux dérivées partielles ayant une structure intéressante. Bull. Soc. Roy. Sci. Liège **25** (1956), 3–4.

Die Verfasserin gibt explizit (in Determinantenform) zwei partielle Differentialgleichungen zweiter Ordnung an, denen die Funktionen

$$z=f_1(a_1X+b_1Y)\cdot f_2(a_2X+b_2Y)$$

bzw.

$$z=F_1(a_1X+b_1Y)+F_2(a_2X+b_2Y)$$

genügen. Das Resultat kann auf Funktionen

$$z=\prod_{k=1}^{n} f_k(G_k),\quad z=\sum_{k=1}^{n} F_k(G_k)$$

verallgemeinert werden ($G_k=a_kX+b_kY$), die Ordnung der entsprechenden partiellen Differentialgleichungen steigt dann auf n [cf. D. S. Mitrinovitch, C. R. Acad. Sci. Paris **237** (1953), 550–551; MR **15**, 317]. *M. Pinl.*

Muller, Mervin E. On discrete operators connected with the Dirichlet problem. J. Math. Phys. **35** (1956), 89–113.

Laplace's equation in N dimensions is approximated by a discrete condition of the form

$$(1)\qquad u(x_0)-\sum_{j=1}^{k} w_j u(x_j)=0.$$

Here x_0, x_j are vectors in the N-space, the "weights" w_j are numbers such that $\sum_{j=1}^{k} w_j=1$, and $x_j=x_0+r_jv_j$, where the r_j are positive scalars and the v_j denote fixed unit vectors with coordinates v_{jl} ($l=1, 2, \cdots, N$). Let $\bar{r}=\max_j r_j$. Under suitable conditions on the w_j, r_j and v_j the left member of (1) is $O(\bar{r}^m)$, $m>2$, as $\bar{r}\to 0$, for all harmonic functions $u(x)$. Then (1) can be regarded as a discrete approximation to Laplace's equation. The author studies numerous conditions for this to be the case. Particular attention is given to the question of finding "optimal" approximations within a given class of approximations of type (1); i.e. approximations for which m has the highest possible value attainable with a given value of k. A typical result in two dimensions is that among the equations (1) using constant weights $w_j=1/k$ the optimal ones are those for which the "neighbor set" x_j forms the vertices of a regular polygon about x_0 as center. The corresponding order of approximation is $O(\bar{r}^k)$. Another theorem states that in N dimensions an approximation which uses $N+1$ points at equal distance r from x_0 is optimal and of order $O(r^3)$ if and only if the weights are equal and the points form a regular simplex whose center is x_0. For computational purposes it is convenient to consider an imbedding of the N-space into an $(N+1)$-space. Then there exists an orthogonal lattice of mesh length h in this $(N+1)$-space such that every lattice space of the N-space posseses an N-dimensional regular neighbor set whose vertices form a regular N-simplex in the lattice with vertices in the N-space. A brief discussion of an interpretation of (1) in terms of a random walk is included. *W. Wasow.*

Pini, Bruno. Teoremi di unicità per problemi generalizzanti i problemi biarmonici fondamentali interno ed esterno. Boll. Un. Mat. Ital. (3) **10** (1955), 465–473.

Let D be a plane region, bounded by a finite number of continuously curved Jordan curves C_j: $\mathbf{x}=\mathbf{x}(s_i)$, s_i denoting the arc length along C_i, $0\leq s_i\leq l_i$, $i=1, 2, \cdots, n$. Suppose that on each C_i three arbitrary functions of class L_2 are defined: $f_i(s_i)$, $g_i(s_i)$ and $h_i(s_i)$. Theorem: If D is bounded, then there exists at most one function $u(\mathbf{x})$ which is biharmonic in D and such that

$$\lim_{t\to+0}\sum_{i=1}^{n}\int_0^{l_i}\Big\{\Big[u(\mathbf{x}(t;s_i))-f_i(s_i)\Big]^2 +\Big[\frac{\partial u}{\partial x}(\mathbf{x}(t;s_i))-g_i(s_i)\Big]^2+\Big[\frac{\partial u}{\partial y}(\mathbf{x}(t;s_i))-h_i(s_i)\Big]^2\Big\}ds_i=0.$$

Thereby $\mathbf{x}(t;s_i)=\mathbf{x}(s_i)+t\mathbf{n}(s_i)$, $\mathbf{n}$ denoting the interior normal unit vector. *A. Huber* (Zurich).

See also: Boreli, p. 1148; Buharinov, p. 1157; Dedecker, p. 1047; Grossman, p. 1139; Kovtun, p. 1079; Morawetz, p. 1149; Yalin, p. 1151.

Difference Equations, Special Functional Equations

Bellman, Richard. On a quasi-linear equation. Canad. J. Math. **8** (1956), 198–202.

The equations

$$(1)\qquad x_i(n+1)=\max_{q_1,\cdots,q_N}\sum_{j=1}^{N} a_{ij}(q_1, q_2, \cdots, q_N)x_j(n)\qquad (x_i(0)=c_i\geq 0),$$

$$(2)\qquad \lambda y_i=\max_{q_1,\cdots,q_N}\sum_{j=1}^{N} a_{ij}(q_1, q_2, \cdots, q_N)y_j\ (i=1, 2, \cdots, N)$$

are examined under the suppositions that the maximums

figuring in the equations are attained in the $(q_1, q_2, \cdots, q_N)$ region considered, that $0<a_{ij}(q_1, q_2, \cdots, q_N)\leqq m$ $(i, j=1, 2, \cdots, N)$ and that the characteristic root $\phi(q_1, q_2, \cdots, q_N)$ of largest absolute value of $\|a_{ij}(q_1, q_2, \cdots, q_N)\|$ assumes its maximum for exactly one [(1)] resp. for at least one [(2)] point of our $(q_1, q_2, \cdots, q_N)$-region. The results are: $\lambda=\max_{q_1,\cdots,q_N}\phi(q_1, q_2, \cdots, q_N)$ is the unique $\lambda>0$ for which (2) has a solution $y_i>0$ (unique up to a multiplicative constant); the solutions of (1) and (2) are connected by the asymptotic relation $\lim_{n\to\infty}[x_i(n)/(y_i\lambda^n)]=a(c_1, c_2, \cdots, c_N)$. Relations of the problem to the theories of dynamic programming and of Markoff processes together with further remarks and possible generalizations are also mentioned in the paper. *J. Aczél* (Debrecen).

Yeh, Kai-yuan. Electric circuit analysis by the method of difference equation. Acta Sci. Sinica 2 (1953), 179–186.

The author writes the linear difference equation of the second order, with constant coefficients, that is satisfied by the steady-state voltage v_x $(x=1, 2, \cdots, n)$ at element x in a ladder network of n elements. After writing the solution that satisfies prescribed terminal conditions in the network, he observes that the corresponding transient voltage $v_x(t)$ is determined by solving a problem of the same type in the Laplace transform $V_x(s)$ of $v_x(t)$.

R. V. Churchill (Ann Arbor, Mich.).

Ghermănescu, M. Sur quelques équations fonctionnelles à deux variables. Acad. R. P. Romîne. Bul. Şti. Secţ. Şti. Mat. Fiz. 7 (1955), 963–975. (Romanian. Russian and French summaries)

The paper is concerned with setting up the totality of solutions of a number of linear functional equations for real functions of two real variables. Of the sixteen results obtained we quote the 6th one as a characteristic example: The totality of measurable solutions of the functional equation $f(x+a, y+b)-f(x, y)=0$, where a and b are given constants, is given by $f(x, y)=F(bx+ay, bx-ay)$, where $F(u, v)$ is an arbitrary measurable function having the period $2ab$ with respect to the variable u.

I. J. Schoenberg (Swarthmore, Pa.).

See also: Sikkema, p. 1068.

Integral Equations, Equations in Infinitely Many Variables

Fel'd, Ya. N. Paired systems of infinite linear algebraic equations, linked with infinite periodic structures. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 215–218. (Russian)

Certain problems in physics lead to a system of equations

$$(1)\quad \sum_{n=0}^{\infty} A_n Z_{m-n}{}^{11}+\sum_{n=-\infty}^{-1} B_n\beta^n Z_{m-n}{}^{12}=C_m{}^{(1)},\quad m\geqq 0;$$

$$\sum_{n=0}^{\infty} A_n\alpha^n Z_{m-n}{}^{21}+\sum_{n=-\infty}^{-1} B_n Z_{m-n}{}^{22}=C_m{}^{(2)},\quad m<0,$$

in which $\{A_n\}$, $\{B_n\}$ are the unknowns. It is shown how, under appropriate conditions, (1) can be led to a nonhomogeneous Hilbert problem for two piecewise holomorphic functions. Let

$$(2)\quad G^{mk}(w)=\sum_{n=-\infty}^{\infty} Z_n{}^{mk}w^n\quad (m, k=1, 2);$$

$$g^+(w)=\sum_{n=0}^{\infty} C_n{}^{(1)}w^n;\quad g^-(w)=\sum_{n=-\infty}^{-1} C_n{}^{(2)}w^n,$$

all series assumed to be convergent in some ring $\varrho_1<|w|<\varrho_2$; and set $f_+(w)=\sum_{n=0}^{\infty} A_n w^n$, $f_-(w)=\sum_{n=-\infty}^{-1} B_n w^n$. Suppose α, β satisfy (2a) $r_1<r_2$ where $r_1=\max(\varrho_1, \varrho_1|\beta|^{-1})$, $r_2=\min(\varrho_2, \varrho_2|\alpha|^{-1})$. The f_+, f_- series are shown to be convergent in $|w|<\varrho_2$, $|w|>\varrho_1$ respectively.

Let L be a contour surrounding $w=0$ and lying in $r_1<|w|<r_2$. System (1) is equivalent to the following relations (a), (b):

$$\text{(a)}\quad \int_L \{f_+(w)G^{11}(w)+f_-(\beta w)G^{12}(w)\}\frac{dw}{w^{n+1}}=\int_{L^-} g^+(\mu)\frac{d\mu}{\mu^{m+1}},\quad m\geqq 0;$$

$$\int_L \{f_+(\alpha w)G^{21}(w)+f_-(w)G^{22}(w)\}\frac{dw}{w^{n+1}}=\int_{L^+} g^-(\nu)\frac{d\nu}{\nu^{m+1}},\quad m<0;$$

$$\text{(b)}\quad \int_L f_+(\alpha^k w)\frac{dw}{w^{n+1}}=0\quad (n<0;\ k=0, 1);$$

$$\int_L f_-(\beta^k w)\frac{dw}{w^{n+1}}=0\quad (n\geqq 0;\ k=0, 1).$$

Here L^+, L^- surround $w=0$ and lie in $\varrho_1<|w|<\varrho_2$, with L^- exterior and L^+ interior, to L. f_+, f_- are then shown to have the form

$$f_\mp(w)=\frac{1}{2\pi i}\left\{\int_{L^-} g^+(\mu)\varphi_\pm{}^{(1)}(w, \mu)d\mu+\int_{L^+} g^-(\nu)\varphi_\mp{}^{(2)}(w, \nu)d\nu\right\},$$

where $\varphi_\pm{}^{(1)}$, $\varphi_\pm{}^{(2)}$ are solutions on L of the Hilbert-type problem

$$\varphi_+{}^{(k)}(w)G^{11}(w)+\varphi_-{}^{(k)}(\beta w)G^{12}(w)=\Phi_-{}^{(k)}(w)(w-\mu)^{k-2},$$

$$\varphi_+{}^{(k)}(\alpha w)G^{21}(w)+\varphi_-{}^{(k)}(w)G^{22}(w)=\Phi_+{}^{(k)}(w)(w-\nu)^{1-k},\quad k=1, 2.$$

Here $\Phi_-{}^{(k)}(w)$, $\varphi_-{}^{(k)}(w)$, $\varphi_-{}^{(k)}(\beta w)$ are functions holomorphic on and exterior to L; $\varphi_-{}^{(k)}(w)$, $\Phi_-{}^{(2)}(w)$ tend uniformly to zero and $\Phi_-{}^1(w)$ is uniformly bounded, as $|w|\to\infty$; $\Phi_+{}^{(k)}(w)$, $\varphi_+{}^{(k)}(w)$, $\varphi_+{}^{(k)}(\alpha w)$ are holomorphic on and interior to L; and $\Phi_-{}^1(\mu)=-1$, $\Phi_+{}^2(\nu)=+1$.

Application of the above treatment is made to a wave problem. *I. M. Sheffer* (University Park, Pa.).

Koelbloed, D. An accurate solution of the integral equation for the Lyman alpha emission in a stationary nebula. Bull. Astr. Inst. Netherlands 12 (1956), 341–348.

Vigranenko, T. I. On solutions of a class of integro-differential equations. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **10** (1953), no. 2, 85–104. (Russian)

The equation

$$U_n y(x)=f(x)+\lambda\int_a^b k(x, t)U_m y(t)dt,$$

where U_n and U_m are differential operators and $K(x, t)$ is a kernel like those in the usual theory of integral equations but with piecewise continuous derivatives of order up to m with respect to x can be reduced to a Fredholm equation. The author discusses the case where $f(x)\equiv 0$ in considerable detail with special attention to the case where λ is an eigenvalue. Formulas for the special case where $K(x, t)$ is degenerate (separable) are given, a

generalization to systems of equations is discussed and several examples considered. *D. C. Kleinecke.*

Bykov, Ya. V. On characteristic values and functions of an integro-differential system. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **10** (1953), no. 2, 55–84. (Russian)

The equation

$$L_n(z) = \lambda p(x)z(x) + \lambda \int_a^b f(x, t) \cdot z(t)dt,$$

where L_n is a self adjoint differential operator, can be changed to

$$Z(x) = \lambda \int k(x, y)p(y)z(y)dy + \lambda \int\int k(x, s)f(s, y)dsdy$$

by operating with the integral operator whose kernel is the Green's function of L_n. If $f(x, t)$ is symmetric and all the functions are in Hilbert space, then this can be written $z=\lambda KLZ$ so that the equation can be treated by the theory of symmetrizable operators [cf. A. C. Zaanen, Linear analysis, Interscience, New York, 1953; MR **15**, 878]. *D. C. Kleinecke* (Albuquerque, N.M.).

See also: Belyakova, p. 1151; Harazov, p. 1114; Rothe and Schmeidler, p. 1061.

Calculus of Variations

Pucci, Carlo. Un problema isoperimetrico per la determinazione della forma di una nave. Atti Accad. Naz. Lincei. Mem. Cl. Sci. Fis. Mat. Nat. Sez. I. (8) **4** (1955), 179–218.

This memoir is a detailed study of a particular isoperimetric problem for curvilinear integrals, by direct methods of calculus of variations. Its main results were summarized in an earlier note [Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **17** (1954), 345–346; MR **16**, 1127]. The problem arose from that of constructing a ship of given tonnage and horizontal cross section of prescribed shape with least surface area.

W. H. Fleming (Lafayette, Ind.).

Adachi, Ryuzo. On the form of a surface of liquid which is in equilibrium under pressure and surface-tension. Kumamoto J. Sci. Ser. A. **2** (1955), 210–212.

Derivation of the equation

$$z_{xx}(1+z_y^2) - 2z_{xy}z_xz_y + z_{yy}(1+z_x^2) + \alpha(1+z_x^2+z_y^2)^{3/2} = 0$$

from a variational condition. *R. Finn.*

See also: Rothe and Schmeidler, p. 1061.

Theory of Probability

Navarro, Sebastian. On a generalization of Pearson curves. Las Ciencias **17** (1952), 435–441. (Spanish)

This is a study of certain surfaces of probability obtained in a form related to the Pearson curves.

E. Frank (Chicago, Ill.).

Blom, Siri. Concerning a controversy on the meaning of 'probability'. Theoria **21** (1955), 65–98.

Ludwig, Otto. Über die stochastische Theorie der Merkmalsiterationen. Mitteilungsbl. Math. Statist. **8** (1956), 49–82.

Survey of the literature on the distribution theory of runs of two or more kinds. Problems involving fixed numbers of elements of the various kinds are treated by combinatorial methods, while problems involving random numbers of elements of the various kinds are treated with the help of difference equations. In the latter case, some new results are obtained. *G. E. Noether.*

Müller, Max. Zur Herleitung des Gaussschen Fehlergesetzes aus der Hypothese der Elementarfehler. Jber. Deutsch. Math. Verein. **58** (1956), Abt. 1, 79–86.

Matsuyama, Noboru; and Takahashi, Shigeru. On the law of the iterated logarithm. Sci. Rep. Kanazawa Univ. **3** (1955), no. 1, 21–26.

Let $\{x_k\}$ $(k=1, 2, \cdots)$ be a sequence of independent random variables and $F_k(x)$ be the distribution function of x_k. Assume that

$$\int_{-\infty}^{\infty} x^2 dF_k(x) = 1, \quad \int_{-\infty}^{\infty} x dF_k(x) = 0.$$

Let further $b_1, b_2, \cdots$ be a sequence of real numbers satisfying $B_n = \sum_{k=1}^n b_k^2 \to \infty$, and

$$|b_n| = o(B_n^{\frac{1}{2}}), \quad b_n = O\left(\left(\frac{B_n \log_2 B_n}{n \log_2 n}\right)^{\frac{1}{2}}\right).$$

The author proves that if there exists a distribution function $F(x)$ for which (c independent of k and r)

$$(1) \qquad \int_{|x|\ge r} dF_k(x) < c \int_{|x|\ge r} dF(x),$$

then the law of the iterated logarithm holds, i.e.

$$\Pr\left[\limsup_{n\to\infty} \frac{\sum_{k=1}^n b_k x_k}{(2B_n \log_2 B_n)^{\frac{1}{2}}} = 1\right] = 1.$$

The authors carry out the proof in detail only if the x_k all have the same distribution, (then (1) trivially holds), but remark that the general case follows by the same method. *P. Erdös.*

Gheorghiu, Şerban. Quelques problèmes concernant la division d'un segment par des points pris au hasard. Acad. R. P. Romîne. Stud. Cerc. Mat. **6** (1955), 243–272. (Romanian. Russian and French summaries)

Let $x_1, \cdots, x_n$ be a sample of n independent observations on a random variable that is uniformly distributed over the interval [0, 1], and let $u_1, u_1+u_2, \cdots, u_1+\cdots+u_n$ be the observations rearranged in ascending order. Let u_{n+1} equal $1-u_1-\cdots-u_n$. The author gives a method of calculating the distribution of $S=\lambda_1u_1+\cdots+\lambda_{n+1}u_{n+1}$, where $\lambda_1, \cdots, \lambda_{n+1}$ are constants. When $0=\lambda_1<\lambda_2<\cdots<\lambda_{n+1}$ he finds the cumulative distribution function of S to be $\sum (\lambda_i - S)^n/P'(\lambda_i)$, where $P(z)$ denotes $z(z-\lambda_2)\cdots(z-\lambda_{n+1})$ and the summation is over values of i for which $\lambda_i - S<0$. [The relevant formula is misprinted on p. 258 of the paper.] He considers next problems "of the second kind": in these $u_1, \cdots, u_{n+1}$ are rearranged in ascending order to give $u_1^*, \cdots, u_{n+1}^*$, and the distribution of a linear function of the latter variables is required. He reduces the problems of the second kind to those of the first, and his results include, as a particular case, formulae for the distribution of $u_1^*+\cdots+u_k^*$ (where $1\le k\le n$). These formulae had been found by

Mauldon [Proc. Cambridge Philos. Soc. **47** (1951), 331–336; MR **12**, 721]. He then considers problems "of the third, fourth, $\cdots$ kinds", which involve further rearrangements, and also certain stochastic processes obtained as limiting cases when $n\to\infty$, the interval [0, 1] being replaced by one of length $O(n)$.

H. P. Mulholland (Birmingham).

Zolotarev, V. M. On analytic properties of stable distribution laws. Vestnik Leningrad. Univ. **11** (1956), no. 1, 49–52. (Russian)

The author reviews some of the results of a previous paper [Dokl. Akad. Nauk SSSR (N.S.) **98** (1954), 735–738; MR **16**, 493] on the interrelations between stable laws of different types and obtains further results and applications. For example, if the characteristic function has logarithm

$$-|t|^\alpha \exp\left\{-i\frac{\pi}{2}\lambda\,\mathrm{sgn}(1-\alpha)t\right\},$$

where $\lambda=1-|1-\alpha|$ and $\alpha\neq1$, and if the corresponding probability density is $p(\cdot,\alpha)$, then

$$\int_0^\infty e^{-su}p(-u,\alpha)du=E_{1/\alpha}(-s)/\alpha.$$

Here E_σ is the Mittag-Leffler function. A corresponding result for the case $\alpha=1$ is obtained, and the bilateral Laplace transforms are also evaluated. In this way an integral evaluation of Pollard [Bull. Amer. Math. Soc. **52** (1946), 908–910; MR **8**, 269] is generalized. *J. L. Doob.*

Freĭman, G. A. An elementary method of proof of limit theorems of the theory of probability. Vestnik Leningrad. Univ. **11** (1956), no. 1, 57–73. (Russian)

The author applies an elementary computational method already applied to number-theoretic problems [Trudy Moscov. Mat. Obšč. **4** (1955), 113–124; MR **17**, 239] to the asymptotic ($k\to\infty$) evaluation of the probability that the sum of k mutually independent integer-valued random variables has a specified value. The expression he derives does not give as good a local limit theorem as that of Prohorov [Dokl. Akad. Nauk SSSR (N.S.) **98** (1954), 535–538; MR **16**, 494], but includes an evaluation of the error term. *J. L. Doob* (Urbana, Ill.).

Lévy, Paul. Le caractère universel de la courbe du mouvement brownien et la loi du logarithme itéré. Rend. Circ. Mat. Palermo (2) **4** (1955), 337–366 (1956).

Proofs and elaboration of results announced in a previous paper [C. R. Acad. Sci. Paris **241** (1955), 689–690; MR **17**, 275]. *J. L. Doob* (Urbana, Ill.).

Gihman, I. I. On a theorem of A. N. Kolmogorov. Kiïv. Derž. Univ. Nauk. Zap. **12** (1953), Mat. Sb. no. 7, 75–94. (Russian)

For each positive integer n, let $\xi_{n1},\cdots,\xi_{nk_n}$ be mutually independent random variables. Let f_1, f_2 be functions on [0, 1], with $f_1<f_2$, $f_1(0)<0<f_2(0)$. Let

$$0=t_{n1}<\cdots<t_{nk_n}=1.$$

Then, under further hypotheses too complicated to state here, the probability that the conditions

$$f_1(t_{nr})<\sum_1^r \xi_k<f_2(t_{nr}),\ r\leq k_n;\ f_1(1)<c<\sum_1^{k_n}\xi_k<d<f_2(1)$$

are fulfilled has the limit $u(0, 0)$. Here $u(t, s)$ satisfies the equation

$$u_t+a(t)u_s+\tfrac12 b(t)u_{ss}+W(t)\left[\int_{f_1(t)}^{f_2(t)}u(t,z)d_z\Phi(t,z-s)-u(t,s)\right]=0$$

in the domain $0<t<1$, $f_1(t)<s<f_2(t)$, with boundary conditions $u(1,s)=1$ on (c,d), $u(1,s)=0$ otherwise, and $u(t,f_1(t))=u(t,f_2(t))=0$ on [0, 1]. The generalization over all previous results, beginning with that of Kolmogorov [Izv. Akad. Nauk SSSR. Otd. Mat. Estest. Nauk (7) **1931**, 959–962] lies in the integral term, which means that the limiting stochastic process is no longer necessarily of diffusion type. *J. L. Doob* (Urbana, Ill.).

Skorohod, A. V. On the limiting transition from a sequence of sums of independent random quantities to a homogeneous random process with independent increments. Dokl. Akad. Nauk SSSR (N.S.) **104** (1955), 364–367. (Russian)

Let S_{ni} be the sum of i mutually independent random variables with a common distribution, let $\zeta_n(t)=S_{n[t(n+1)]}$, for $0\leq t<1$, and suppose that S_{nn} has a limiting distribution when $n\to\infty$. This distribution is then infinitely divisible and determines a stochastic process $\{\zeta(t), 0\leq t\leq1\}$ with stationary independent increments. Let K be the space of functions defined on [0, 1], with limits from the left and right at all points. Then the $\zeta(t)$ process can be assumed to have its sample functions in K. If f is defined on K, the author finds a condition on f such that the distribution of $f[\zeta_n(\cdot)]$ converges weakly to that of $f[\zeta(\cdot)]$. The condition is that, in an appropriate topology, f be continuous almost everywhere on K, where the K measure is that of the $\zeta(t)$ process. This result is proved using the special case, due to Donsker [Mem. Amer. Math. Soc. no. **6** (1951); MR **12**, 723] in which the $\zeta(t)$ process is the Brownian-motion process. In this way, the asymptotic limit distributions of such random variables as

$$\sup_{1\leq i\leq n} S_{ni},\quad \frac1n\sum_1^n S_{ni}{}^k,\quad S_{nn}/\sup_{1\leq i\leq n}(S_{ni}-S_{ni-1})$$

can be identified as the corresponding distributions on the $\zeta(t)$ process. In the paper reviewed above Gihman has obtained a similar result for f the characteristic function of a K set determined by the inequality $\phi_1(t)\leq x(t)\leq\phi_2(t)$, where ϕ_1, ϕ_2 are specified and satisfy certain conditions.

J. L. Doob (Urbana, Ill.).

Takahashi, Shigeru. On the convergence of some random Riemann-sums. Sci. Rep. Kanazawa Univ. **4** (1955), no. 1, 29–34.

Let R be the real numbers, let the Borel function $(f(t): t\in R)$ have period 1 and be summable on $I=[0,1]$, let the random variables $t_1, t_2, \cdots$ be independent and uniformly distributed on I, let $s\in R$, and consider the Riemann sum S_n associated with the function

$$(f(t+s): t\in I)$$

and the partition $0, t_1, t_2, \cdots, t_n, 1$. The author shows that $E(|S_n-\int_I f(t)dt|)=o(1)$ $(n\uparrow+\infty)$ and that if, for some $\varepsilon>0$,

$$\int_I |f(t+h)-f(t)|dt=O(1/|\log h|^{1+\varepsilon})\quad (h\downarrow 0),$$

then $\Pr(S_n\to\int_I f(t)dt,\ n\uparrow+\infty)=1$, provided s is not contained in some exceptional set of measure 0.

H. P. McKean, Jr. (Princeton, N.J.).

Fisz, M. The realizations of some purely discontinuous stochastic processes. Bull. Acad. Polon. Sci. Cl. III. 4 (1956), 63–65.

Let I_0 be a compact interval, and let $\{x_t, t \in I_0\}$ be a separable stochastic process. It is supposed that, for each $t \in I_0$, almost all sample functions are continuous at t. Let $\alpha(I)$ be the probability that a sample function does not have the same value at the endpoints of the interval I, and let $A(I_0)$ [$\bar{A}(I_0)$] be the [upper] Burkill integral of α on I_0. The process is said to be purely discontinuous if almost every sample function is constant in some neighborhood of each of its continuity points. The following theorems are stated without proof. (1) If $\bar{A}(I_0)<\infty$, then (i) the process is purely discontinuous, (ii) $\infty>\bar{A}(I_0)=A(I_0)=$the expected number of sample function discontinuities, (iii) $\lim \alpha(I)/|I|=Q(t)$ (as I shrinks to t) exists for almost all t, and its integral over I_0 is at most $A(I_0)$. (2) If α is an absolutely continuous interval function, then $\bar{A}(I_0)<\infty$, and the inequality just described becomes an equality. (3) If the process is purely discontinuous and if the expected number of discontinuities on I_0 is finite, then $\bar{A}(I_0)<\infty$.

Dobrušin has also obtained a condition [Mat. Sb. N.S. 34(76) (1954), 541–556; MR 16, 150] sufficient that the second hypothesis of (3) be satisfied, for a Markov process with finitely many states.

J. L. Doob.

Watanabe, Yoshikatsu. Aufgaben betreffend das Irrfahrtproblem. J. Gakugei Tokushima Univ. Nat. Sci. Math. 6 (1955), 41–49.

Pólya [Math. Ann. 84 (1921), 149–160] proved that a point moving randomly on a d-dimensional lattice returns to the origin infinitely often with probability 1 for $d=1$ or 2, while for larger d it is certain to wander off to infinity. For their bearing on this result, the author examines three further cases. In the first the square lattice in the plane is replaced by a lattice of triangles so that at each step there are six (equiprobable) possibilities. This case has been mentioned by Dvoretzky and Erdös [Proc. 2nd Berkeley Symposium Math. Statist., 1950, Univ. of California Press, 1951, pp. 353–367; MR 13, 852] and a similar lattice but with one way paths has been treated by Lehman [ibid., pp. 263–268; MR 13, 363]. It is shown that the probability of return to the origin in m steps is asymptotically equal to $3^{\frac{1}{2}}/2\pi m$ and hence the behavior with respect to return to the origin is exactly that of the plane. In the second case, two plane lattices are joined together by risers at every lattice point (a rectangular network of streets is joined to a similar underground network by elevators at every street crossing); here the behaviour is shown to be that of a 3-dimensional space lattice, though at each step there are five possibilities instead of the six for the space lattice. Finally Pólya's result is re-examined adding the possibility of no movement at each step (this has probability p and the remaining probability $1-p$ is divided equally among the $2d$ paths); Pólya's conclusions are unchanged.

J. Riordan (New York, N.Y.).

Wigner, Eugene P. Characteristic vectors of bordered matrices with infinite dimensions. Ann. of Math. (2) 62 (1955), 548–564.

Considerations, in part heuristic, on eigenvalue and -vector distributions for certain special classes of random matrices, arising from quantum mechanical investigations.

J. G. Wendel (Ann Arbor, Mich.).

Gani, J. Some problems in the theory of provisioning and of dams. Biometrika 42 (1955), 179–200.

It is pointed out that the problems in provisioning considered by Pitt [J. London Math. Soc. 21 (1946), 16–22; MR 8, 281] and those in the theory of dams considered by Moran [Austral. J. Appl. Sci. 5 (1954), 116–124, MR 16, 269] can be treated as special cases of the general problem of studying a storage function

$$S(t)=I(t)-D(t)-F(t)$$

where I, D and F refer to the input, output and overflow respectively. Pitt's results are summarized and applied to give a solution to a problem in the theory of dams, and Moran's results similarly applied to provisioning. Numerical examples are included.

D. V. Lindley.

Moran, P. A. P. A probability theory of dams and storage systems: modifications of the release rules. Austral. J. Appl. Sci. 6 (1955), 117–130.

The probability theory of dams developed in a previous paper [same J. 5 (1954), 116–124; MR 16, 269], is extended in several ways: (i) by considering modifications of the release rule; (ii) by considering different release rules for different months of the year; (iii) by obtaining an exact solution for a negative exponential input; (iv) by devising a shorter numerical method for obtaining an approximate solution in the case of a type 3 input.

Author's summary.

Zoroa Terol, Procopio. A minimum problem occurring in architecture. Trabajos Estadist. 6 (1955), 197–208. (Spanish. English summary)

A problem about the most convenient placing of the street-door of a long building (for given expectations of traffic through internal doors) is reduced by the author to a special case of the following problem: to minimize

$$S(z)=(1-p)\int_{-\infty}^{z}(z-x)dG(x)+p\int_{z}^{\infty}(x-z)dH(x),$$

where $G(x)$ and $H(x)$ are cumulative distribution functions and $0<p<1$. He finds easily that $S(z)$ is smallest where $(1-p)G(z)+pH(z)-p$ vanishes (or changes sign).

H. P. Mulholland (Birmingham).

Bertaut, E. F. La méthode statistique en cristallographie. I. Acta Cryst. 8 (1955), 537–543.

The author employs the Dirac δ-function to rederive the Fourier integral theorem which in turn is used to obtain the joint probability distributions of several crystal structure factor $F_{h_i k_i l_i}$. The latter are defined by means of

$$F_{hkl}=\sum_{j=1}^{N} f_{jhkl}\exp\{2\pi i(hx_j+ky_j+lz_j)\}$$

where the f_{jhkl} are the (known) atomic scattering factors and x_j, y_j, z_j are the coordinates of the jth atom. The probability distributions are expressed by means of infinite series, the separate terms of which depend on the nature of the relationships among the relevant indices h_i, k_i, l_i and on the symmetry of the crystal structure (i.e. the space group).

Reviewer's note: The method used here was initiated by the reviewer and Karle [Amer. Cryst. Assoc. Monograph no. 3 (1953)] who obtained essentially the same results as those described in the present paper without, however, employing the Dirac δ-function.

H. A. Hauptman (Washington, D.C.).

Bertaut, E. F. La méthode statistique en cristallographie. II. Quelques applications. Acta Cryst. 8 (1955), 544–548.

By means of the joint probability distributions of several structure factors $F_{h_i k_i l_i}$, probable relationships among the structure factors, which depend upon the space group symmetries, are established. If the crystal structure has a center of symmetry, the only case considered here, the structure factor F_{hkl} is real. The determination of the F_{hkl}'s is of fundamental importance since they are the coefficients of the 3-dimensional Fourier series which yields the electron density in the crystal. However, only the magnitudes of the F_{hkl} are determined from experiment. Hence the ultimate purpose of the structure factor relationships is to determine the signs of the F_{hkl} when only their magnitudes are initially known. The approach here described was initiated by the reviewer and Karle in their Monograph [Amer. Cryst. Assoc. Monograph no. 3 (1953)] which also contains most of the results of the present paper. The chief contribution of the author is to list explicitly several higher order terms in the series expansions describing the joint probability distributions.

H. A. Hauptman (Washington, D.C.).

Feinstein, Amiel. A new basic theorem of information theory. Research Laboratory of Electronics, Massachusetts Institute of Technology, Tech. Rep. No. **282** (1954), i+28 pp.

The author reformulates the Shannon theorem on the relation between source entropy and channel capacity, making it possible to give a rigorous and intuitively satisfying treatment. Although his treatment is incomplete in some ways, the material has now been reworked, generalized and elaborated by Hinčin [see the following review].

J. L. Doob (Urbana, Ill.).

Hinčin, A. Ya. On the basic theorems of information theory. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1(67), 17–75. (Russian)

The theorems of information theory on the relation between the capacity of a noisy channel and entropy of a source have been rather confused, since Shannon's pioneering work [Bell System Tech. J. **27** (1948), 379–423, 623–656; MR **10**, 133] because neither statements nor proofs have been formulated in precise language. McMillan [Ann. Math. Statist. **24** (1953), 196–219; MR **14**, 1101] refined Shannon's concepts and generalized some of his results. Feinstein [Trans. I. R. E. Professional Group on Information Theory no. **PGIT-4** (1954), 2–22] reformulated the relationship in question but gave a rather sketchy treatment. Hinčin has now written a full treatment, sometimes sacrificing generality to preserve an elementary style. The derivation is so complete in itself that it even includes the proof of a needed martingale convergence theorem, and in fact about the only theorem which is used but not proved is the ergodic theorem. All alphabets are supposed finite. If a source has entropy H per symbol, and is the input of a channel, the speed of transmission is defined as H less the input entropy per symbol relative to the channel output. The ergodic transmission capacity of a stationary channel is defined as the supremum over all stationary ergodic inputs of the transmission speeds. In the following, it is supposed that there is given (1) a stationary channel with finite memory m, no foresight, and ergodic transmission capacity C (2) a stationary ergodic source, not necessarily with the same alphabet as the channel input alphabet, of entropy $H_0<C$. The source is coded into the channel in such a way that each sequence of n letters of the source alphabet (called an α sequence below) is coded into a sequence of $m+n$ letters of the channel input alphabet. These $m+n$ letters yield letters of the output alphabet, the last n of which will be called a β sequence. To each β sequence is made to correspond the most probable α sequence, given that β sequence. It is shown that, if $\varepsilon>0$, then, for sufficiently large n, the coding can be done in such a way that, if an α sequence is chosen in accordance with source probabilities, is coded, sent through the channel, and identified in terms of the corresponding β sequence as just explained, then the probability of identification error is at most ε. Moreover it is shown that the coding can be done in such a way that the transmission speed (in terms of a composite channel with input the source output) is arbitrarily near H_0. The author is unable to prove the impossibility of conclusions like these when $H_0>C$, although, in less careful treatments such a proof appears trivial. The difficulty is linked with the fact that C is defined in terms of ergodic stationary sources rather than arbitrary stationary sources.

J. L. Doob (Urbana, Ill.).

Tortrat, Albert. Sur la définition de l'entropie en théorie de l'information. Ann. Télécommun. **10** (1955), 39–47.

The author considers the problems inherent in trying to find a satisfactory valid general definition of entropy in the theory of information for both the discontinuous and continuous cases. A mathematical appendix on the development of functions in the time domain, and in the frequency domain, is the basis for a definition of entropy for messages depending on a denumerable infinity of parameters.

S. Kullback (Washington, D.C.).

Faddeev, D. K. On the concept of entropy of a finite probabilistic scheme. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 1(67), 227–231. (Russian)

The author shows that the entropy H of a system of probabilities $p_1, \cdots, p_n$ is given by $-c\sum p_i \log p_i (c>0)$ if the following axioms, a modification of those given by Shannon [Bell System Tech. J. **27** (1948), 379–423, 623–656; MR **10**, 133], are satisfied. (1) If $n=2$, H is continuous, and is positive for at least one pair of argument values. (2) H is symmetric. (3) If $n\geqq 2$, and if $p_n=q_1+q_2$, then

$$H(p_1, \cdots, p_{n-1}; q_1, q_2)=H(p_1, \cdots, p_n)+p_n H(q_1/p_n, q_2/p_n).$$

J. L. Doob (Urbana, Ill.).

Laloë, Michel. Information, sélection, information sélective. La sécurité est-elle une constante d'univers? Ann. Télécommun. **10** (1955), 31–38.

The author concludes that three different physical notions, "information", "selection" and the classic "selective information" have been confused. Information corresponds to entropy; selection is an operation which consumes information; selective information formally but not physically corresponds to entropy. The author considers that the knowledge of an event with probability p is equivalent to the quantity of information

$$I=\log_e [p/(1-p)].$$

He arrives at this definition by applying results of Woodward [Probability and information theory···, McGraw-Hill, New York, 1953; MR **15**, 450] to the following problem: A desires to inform B, over a telecommunication

channel, that a certain event has occurred. Suppose A is accountable for the energy which he transmits over the channel from the instant he is aware of the event but not for the energy used prior to this instant nor for the energy B must use to assure reception. How can A do this most economically? The selection problem is considered to be the following. What is the risk of error to which B is subject in making a decision influenced by the signalled event? The problem the author considers regarding selective information is the following: A telecommunication channel is given with some fixed band pass and peak power. The cost of a message is the time the channel is used for its transmission. How can the most information be transmitted for the least cost?

S. Kullback.

Schützenberger, M. P. Contribution aux applications statistiques de la théorie de l'information. Publ. Inst. Statist. Univ. Paris **3** (1954), no. 1–2, 3–117.

This memoir consists of three parts, each of which is meant to be a self contained unit. The first part is an exposition of concepts and results of lattice theory, for the most part classical material that may be found in G. Birkhoff, "Lattice theory" [Amer. Math. Soc. Colloq. Publ. v. 25, rev. ed., New York, 1948; MR **10**, 673] or in V. Glivenko, „Théorie générale des structures" [Hermann, Paris. 1938]. Applications are made to a problem on order statistics and the distributions of particles into cells. The latter had been presented earlier by the author [C. R. Acad. Sci. Paris **232** (1951), 1805–1807; MR **13**, 51]. The second part, Information Theory, presents on axiomatic grounds a general definition of an information and considers its interrelationship with other measures of information and various statistical applications. The third part, Principles of Grouping, considers a mathematical model which represents certain systems of objects such that a single observation may be capable of characterizing many of the objects, for example, the observation of the product of a sequence of numbers permits the inference that all are different from zero or that at least one of the numbers is zero. The concept of a locally optimum procedure is introduced as a substitute for the optimum solution, usually unavailable because of combinatorial complexity. Examples are given of applications to problems in estimation, hypothesis testing and genetics.

The author defines an information as the mean value, over the set of states, of the resultant of applying a linear operator to the logarithm of the a priori probability of each state (probability density in the case of continuous variables). The operator must be such that the corresponding information is non-negative. Information is additive for independent random variables. The connection with the information of Shannon-Wiener and that of Fisher had been announced earlier by the author [ibid. **232** (1951), 925–927; MR **12**, 623]. The author defines the information $W(\theta)$ of Wald (so named because of the extensive use by Wald of the logarithm of the likelihood ratio in Sequential Analysis) by the linear operator $[\ ]_{\theta=\theta_0}^{\theta=\theta_1}$, where θ_i, $i=0$, 1, are the parameters characterizing the distributions under the hypotheses $H_0:\theta=\theta_0$ and $H_1:\theta=\theta_1$. There are two values

$$W(\theta_1)=E\left(\log\frac{f(x,\theta_1)}{f(x,\theta_0)}\,\middle|\,\theta=\theta_1\right) \text{ and }$$

$$W(\theta_0)=E\left(\log\frac{f(x,\theta_0)}{f(x,\theta_1)}\,\middle|\,\theta=\theta_0\right).$$

For the binomial the author shows that

$$2(p_0-p_1)^2+(4/3)(p_0-p_1)^4=p_0\log(p_0/p_1)$$
$$+q_0\log(q_0/q_1)\leq(p_0-p_1)^2/2pq,$$

for $p_i+q_i=1$, $q_0>p_0$, and $p=p_0$ (or p_1) according as $(q_0-p_0)>(\text{or }<)(q_1-p_1)$. The author defines a "combinatorial" information (information de tri) of the first kind by the operator $[t\partial/\partial t]|_{t=0}$, for a family of elementary events whose a priori probabilities are zero for the parameter $t=0$ and developable as a series of increasing powers of t. This is related to problems concerning the number of elementary events which have occurred under certain conditions. An information of the second kind is associated with the operator $[\ ]_{t=0}$ which yields the logarithm of the number of ways in which the occurrence of n elementary events imply the occurrence of an observed event. Chi-square, defined by

$$\chi^2=\sum_i (p_i(\theta_0)-p_i(\theta_1))^2/p_i(\theta_0),$$

has certain of the properties of an information but is not additive for the composition of independent distributions, and is therefore classed by the author as a pseudo information. In connection with a discussion of the role of cumulants as information the author introduces the probability $q_i=p_i(\varphi(t))^{-1}\exp(\sqrt{(-1)}tx_i)$, where $p_i=\Pr(\zeta=x_i)$ and $\varphi(t)=\sum_i p_i\exp(\sqrt{(-1)}tx_i)$. The reviewer does not understand the interpretation of the q_i, which are in general complex, as probabilities. The reader is cautioned to look out for misprints. On page 65 the results for a and b should read

$$a=\lim_{r\to+\infty}\sup r^{-1}\log|\varphi(\sqrt{(-1)}r)|,$$

$$b=\lim_{r\to+\infty}\sup r^{-1}\log|\varphi(-\sqrt{-(1)}r)|.$$

To make the number listing of the references in the bibliography of part two on pages 67–69 consistent with the references thereto in the text itself, subtract one from the listing number for items 22 to 93 inclusive.

S. Kullback (Washington, D.C.).

Have

★ **Brillouin, Leon. Science and information theory.** Academic Press Inc., New York, 1956. xvii+320 pp. $6.80.

The author is one of the leading members of an increasing group of enthusiasts who have attempted to unify a number of diverse physical and engineering fields under the banner of the expression $-\sum p_i\log p_i$. The present work is based on a series of lectures designed for an engineering audience. As such it assumes little previous knowledge beyond elementary physics, and the mathematical discussions are largely heuristic.

The first six chapters are devoted, for the most part, to an exposition of portions of Shannon's theory of communication, i.e. coding, and its extensions and applications by other workers, such as Mandelbrot, Hamming, and the author. Among the problems considered are language statistics, error correcting codes, and filing. It should be noted that the definition of channel capacity, on page 29, is somewhat more restrictive than Shannon's.

Chapter 8 deals with Fourier analysis, and "sampling" theorems.

The next portion of the book, chapters 9–18, starting with "Summary of Thermodynamics", is most closely related to the author's own work in the field. The climax is reached with the establishment of the negentropy principle of information, generalizing Carnot's principle,

in chapter 12. This is followed by detailed analyses of Maxwell-type demons in a variety of models. Chapter 18 is entitled, "Writing, Printing, and Reading". Here, as well as at a number of other places, some metaphysics crops up. As the author points out, there are still some problems whose formulation is incomplete.

Chapter 19 treats numerical analysis and computers, and is actually somewhat separate from the other material. In the sections devoted to stability of numerical methods it might have been worth while to append more references to the extensive literature on difference and difference-differential equations.

Chapter 20 closes with a discussion of miscellaneous problems, such as organization, information in a physical law, and the problem of semantic information. There is also a section on "Examples of Problems Beyond the Present Theory".

This book will prove extremely valuable to anyone who wishes to familiarize himself with the ideas of the young science of information theory. *E. Reich.*

Have

★ **Zadeh, L. A. General filters for separation of signal and noise.** Proceedings of the symposium on information networks, New York, April, 1954, pp. 31–49. Polytechnic Institute of Brooklyn, Brooklyn, N.Y., 1955.

Let r be a time function from which it is desired to extract a signal s in the presence of a noise n by means of a filter F. The author presents an exposition of the conceptual solution of this problem from the point of view of decision theory.

Section 6 discusses perfect filters. If R, S, N are the spaces of all r, s, and n, respectively, then, under certain restrictions, the possibility of perfect filtering implies dimension rate $R=\dim$ rate $S+\dim$ rate N. *E. Reich.*

★ **Rosenbloom, Arnold. Analysis of linear systems with randomly time-varying parameters.** Proceedings of the symposium on information networks, New York, April, 1954, pp. 145–153. Polytechnic Institute of Brooklyn, Brooklyn, N.Y., 1955.

The author considers the behavior of systems governed by the differential equation $Le_0(t)+e_0(t)=e_i(t)$ for the cases (a) $L=(1/K(t))d/dt$, (b) $L=(d/dt)1/K(t)$, and (c) $L=d/dt$, e_i and e_0 represent input and output voltages respectively, when $K(t)$ is a Gaussian random process. When $K(t)$ is stationary and $e_i(t)$ is a unit-step function, then $e_0(t)$ has a logarithmico-normal marginal distribution. *E. Reich* (Santa Monica, Calif.).

Slepian, David. A class of binary signaling alphabets. Bell System Tech. J. **35** (1956), 203–234.

Consider the Boolean ring B generated by n atoms, and its natural metric, $d(A, B)=d(A\cup B)-d(AB)=d(A+B)^n$, where $d(X)$ is the lattice dimension function and $A+B$ is, of course, the symmetric difference $A+B=AB'\cup A'B$. In an n binary digit coordinatization each element A is called a letter and $d(A)$ is the number of coordinates equal to one (also called the weight of A). The author defines an (n, k) group alphabet to be an additive subgroup of B_n of order $\mu=2^k$ represented in the fixed coordinatization. The group alphabets are then shown to be the same as the "systematic codes" in the sense of R. W. Hamming [same J. **29** (1950), 147–160; MR **12**, 35].

Now let a system of representatives, $I, S_1, S_2, \cdots, S_\nu$ of the cosets of an (n, k) group alphabet, $I, A_1, A_2, \cdots, A_\mu$, be so chosen that each S_i has minimum dimension (weight) in its coset $(\nu=2^{n-k})$; note that it may be impossible further to restrict the choice of representatives to form a subgroup.

Also let a "noisy communication channel", in which only elements of a given group alphabet are transmitted, but any elements of B_n might be received, employ the following detection method: If the received signal, B, belongs to the coset with representative S, the detector prints $A=S+B$. If the communication channel has the probability p of sending a bit incorrectly and $q=1-p$ of sending one correctly, where $0\leq p\leq\frac{1}{2}$, the author shows the detection scheme to be a maximum likelihood detector.

Finally, by application of the theory of group representations the author constructs tables for best group alphabets and their parity check rules for most of the pairs (n, k) where $4\leq n\leq 12$ and $2\leq k\leq n-2$. *S. Gorn.*

★ **Komamiya, Yasuo. Application of logical mathematics to information theory. (Application of theory of groups to logical mathematics.)** Proceedings of the Third Japan National Congress for Applied Mechanics, 1953, pp. 437–442. Science Council of Japan, Tokyo, 1954.

The author states that, if m_0 is the maximum number of binary words of length n for which any pair has distance at least p in the sense of R. W. Hamming [Bell System Tech. J. **29** (1950), 147–160; MR **12**, 35], then $2\leq m_0\leq 2^{n-p+1}$ and m_0 is a power of two.

The second statement contradicts the following theorem due to M. Plotkin [Master's Thesis, Univ. of Pennsylvania, 1952]: If m is an integer such that $4m-1$ is prime, and $n=4m$ and $p=2m$, then $m_0=8m$.

Hamming's minimum distance codes all have a number of elements equal to a power of two because of his requirement that they be "systematic". [See the paper reviewed above.] *S. Gorn* (Philadelphia, Pa.).

De Francesco, Silvio. Note intorno al teorema di Shannon. Ann. Geofis. **7** (1954), 195–207.

The theorem in question given by Shannon [Proc. I.R.E. **37** (1949), 10–21; MR **10**, 464] is: If a function $f(t)$ contains no frequencies higher than W cps, it is completely determined by giving its ordinates at a series of points spaced $1/2W$ seconds apart. The author demonstrates this theorem as a case of complete convergence of a generalized Fourier expansion basing his results on the theorem: If $\int_{-\infty}^{\infty}|f(t)|dt$ converges and $F(\omega)=(2\pi)^{-1}\int_{-\infty}^{\infty}f(t)e^{-i\omega t}dt$ vanishes outside the interval $(-2\pi W, 2\pi W)$, where W is finite and positive, then

$$f(t)=\int_{-\infty}^{\infty}f(\tau)\frac{\sin 2\pi W(t-\tau)}{(t-\tau)}d\tau.$$

Under suitable restrictions, the inherent error is given as

$$\varepsilon=(2\pi)^{-1}\int_{-2\pi W}^{2\pi W}F(\omega)\left(1-\frac{(\omega/4W)\exp(-i\omega/4W)}{\sin(\omega/4W)}\right)e^{i\omega t}d\omega.$$

S. Kullback (Washington, D.C.).

Leonov, Yu. P. On filtering nonstationary random functions. Avtomat. i Telemeh. **17** (1956), 97–106. (Russian)

Following Blanc-Lapierre [Blanc-Lapierre and Fortet, Théorie des fonctions aléatoires, Masson, Paris, 1953; MR **15**, 883], the author considers processes which can be written as Fourier transforms of other processes, in the form $\int_{-\infty}^{\infty}e^{2\pi i\nu t}dx(\nu)$. If $X_1(t)=S(t)+N(t)$ (signal plus

noise) defines a process which is the sum of two such processes, he wishes to choose $R(t)$ in such a way that $\lim_{T\to\infty}(2T)^{-1}\int_{-T}^{T}E\{|e(t)|^2\}dt$ is minimized, where $e(t)=X_D(t)-X_R(t)$, $X_R(t)=\int_0^\infty R(\tau)X_1(t-\tau)d\tau$, and $X_D(t)$ is the result of some linear operation on the $S(t)$ process. (The corresponding problem without the denominator T in the integral to be minimized is also treated.) The function R must satisfy a Wiener-Hopf integral equation, and the gain corresponding to the solution (Fourier transform of R) is evaluated explicitly. *J. L. Doob* (Urbana, Ill.).

Čencov, N. N. Wiener random fields depending on several parameters. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 607–609. (Russian)

Let $\{x(s, t), 0\leq s, t\leq 1\}$ be a separable stochastic process with the following properties. (a) For $s=0$ or $t=0$ the sample functions are almost all continuous. (b) The second difference of a sample function over a rectangle is a normally distributed random variable with mean 0 and variance the area of the rectangle. (c) Such second differences over non-overlapping rectangles are mutually independent. It is proved that almost all sample functions of the process are continuous, and it is stated that the n-dimensional case can be treated similarly. *J. L. Doob.*

See also: Blanc and Liniger, p. 1137; Santaló, p. 1059; Ville, p. 1044.

Mathematical Statistics

Have

★**van der Waerden, B. L.; und Nievergelt, E. Tafeln zum Vergleich zweier Stichproben mittels X-Test und Zeichentest. Tables for comparing two samples by X-test and sign test.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. iv+34 pp. DM 4.80.

The X-test for the two-sample problem suggested by the first author [Math. Ann. **126** (1953), 93–107; MR **15**, 46] is based on the statistic $X=\sum_1^g \psi(r/(n+1))$, where $\psi(u)$ is the inverse of the standard normal cdf and r stands for the ranks of the g observations in the first sample in the overall ranking of the $g+h=n$ observations in both samples. Under the hypothesis that both samples have been drawn from the same population, X is asymptotically normally distributed with mean 0 and variance $ghQ/(n-1)$ where $Q=n^{-1}\sum_{i=1}^n \psi^2(i/(n+1))$. The following tables are provided: Table 1. Some frequently used significance points of the standard normal distribution; Table 2. Values (2D) of $\psi(r/(n+1))$, $r=1(1)n$, $n=6(1)50$; Table 3. Upper significance points (2D) for X for significance levels .025, .010, .005, $|g-h|\leq 5$, $n=6(1)50$; Table 4. Values (2D) of $\psi(u)$, $u=.000(.001).999$; Table 5. Values (3D) of Q, $n=1(1)150$; Table 6. Upper and lower limits for the sign-test, two-sided significance levels .05, .02, .01, $n=5(1)100$, where n is the total number of + and − signs.

A 14-page introduction discusses theoretical aspects of the X- and sign-tests and describes how the tables have been computed. Five pages (also in English) give instructions for the use of the tables, as well as examples. *G. E. Noether* (Boston, Mass.).

Trickett, W. H.; Welch, B. L.; and James, G. S. Further critical values for the two-means problem. Biometrika 43 (1956), 203–205.

Letting y be a normally distributed estimate of a population parameter η with sampling variance $\lambda_1\sigma_1^2+\lambda_2\sigma_2^2$, where λ_1 and λ_2 are known positive constants, and s_1^2 and s_2^2 are estimates of σ_1^2 and σ_2^2 distributed in the standard fashion with ν_1 and ν_2 degrees of freedom respectively, the authors have calculated the one-sided 2.5% and .5% points of $v=(y-\eta)/\sqrt{(\lambda_1 s_1^2+\lambda_2 s_2^2)}$ for all combinations $\nu_1=8, 10, 12, 15, 20, \infty$, $\nu_2=$same values, $\lambda_1 s_1^2/\lambda_1 s_1^2+\lambda_2 s_2^2=0(.1)1$. Previously the one-sided 5%, 1% points of v were given by Aspin [Biometrika 36 (1949), 290–293; MR **11**, 527]. *L. A. Aroian.*

Petrov, V. V. Corrections to the paper, "On the method of least squares and its extremal properties". Uspehi Mat. Nauk (N.S.) **11** (1956), no. 2(68), 250–251. (Russian)

See Uspehi Mat. Nauk **9** (1954), no. 1(59), 41–62; MR **15**, 971.

Have

★**Breny, H. A propos de la méthode de Daniels pour l'échantillonnage des faisceaux de fibres parallèles.** Colloque sur l'analyse statistique, Bruxelles, 1954, pp. 177–186. Georges Thone, Liège; Masson & Cie, Paris, 1955.

This is mainly a mathematical discussion of Daniels' model [Palmer and Daniels, J. Text. Inst. Trans. Sect. **38** (1947), 94–101; Daniels, ibid. **33** (1942), 137; Suppl. J. Roy. Statist. Soc. **5** (1938), 89]; in which both the length of a fibre (l) and the position (t) of its head/tail are distributed statistically. Various statistical constants for a sample from such a bundle are deduced and a numerical example is given for a log-normal distribution of l. *M. E. Wise* (Penarth).

Poti, S. J. Measures of over-all efficiency of sample multinomial tables. Calcutta Statist. Assoc. Bull. 6 (1955), 102–112.

Consider a multinomial distribution. The author is interested in comparing various sampling plans in order to "best" estimate the unknown parameters. Actually the discussion is limited to using a numerical example to compare a systematic sampling plan to a random sampling plan. No concrete conclusions can be drawn because of the heuristic nature of the paper. *M. Muller.*

Bracewell, R. N. Correcting for running means by successive substitutions. Austral. J. Phys. 8 (1955), 329–334.

Consider a function $f(x)$. Let the average of f over an interval of length ξ centering at x be denoted by $g(x)$. For the purpose of estimating f, given g, one may average g over intervals of length ξ, note the effect of this process, and apply the negative of this effect to g. Next, by iteration of this process a sequence of estimates of f is formed. The conditions of Bracewell and Roberts for the convergence of this sequence (especially convergence to f) are quite stringent. Despite this fact the author notes that the first member or two of the sequence may be for some purposes a substantial improvement over g for estimating f. He traces this circumstance to the fact that improvement and deterioration in the sequence are taking place simultaneously in different parts of the spectrum of the Fourier transforms of the approximating functions. If the bands in which the deterioration is commencing are of lesser importance to us, the early members of the sequence may be substantially better than g as estimates of f. The transformation improves the approximation in the band of lowest frequency, which is often that of greatest concern. *A. Blake* (Royal Oak, Mich.).

Kivéliovitch, M.; et Vialar, J. Quelques nouveaux tests pour l'étude des séries chronologiques III. J. Sci. Météorol. 7 (1955), 259–271 (1 plate).

Given a sequence of real numbers confined to n discrete values, $E_0<E_1<\cdots<E_n$. Consider any values a_1, a_2, $\cdots$, a_n intermediate between these, that is, such that $E_{i-1}<a_i<E_i$ for each i. The author works out confidence limits for the frequencies with which the function crosses the various a_i, and illustrates the method with various examples, chiefly from meteorology. It is remarkable in these examples how effectively these easily computed statistics discover various departures from randomness in the data. *A. Blake* (Royal Oak, Mich.).

Weiller, A. R. Réflexions sur les tests du hasard de MM. Kiveliovitch et Vialar. J. Sci. Météorol. 7 (1955), 295–299.

The author constructs an example to show that a value of one of Kiveliovitch's and Vialar's order statistics can have been derived from either of two populations that differ slightly in their trend. This is a question of power of the test. In addition to this point it is noteworthy that a statistic does not ordinarily test a population for randomness, but only for absence of a particular type of departure from randomness. *A. Blake* (Royal Oak, Mich.).

Takashima, Michio. Tables for testing randomness by means of lengths of runs. Bull. Math. Statist. 6 (1955), no. 1–2, 17–23.

Suppose that there are elements of two kinds A, and B. Let the number of A's and B's be m and n respectively. Let $Q_1(t)$ denote the probability that there appears at least one A-run of length t or longer. Let $Q(t)$ be the probability that there appears at least one A- or B-runs of length t or longer. Define t_5 and t_1 ($\bar{t}_5$ and $\bar{t}_1$) as the smallest integers t which make $Q(t)$ ($Q_1(t)$) less than 0.05 and 0.01 respectively. The purpose of the present note is to give more comprehensive tables of t-values. Tables of t_5, t_1 ($\bar{t}_5$, and $\bar{t}_1$) are given for every pair (m, n) where $1\leqq m\leqq 25$ and $1\leqq n\leqq 25$. *M. Muller* (New York, N.Y.).

Darwin, J. H. The behaviour of an estimator for a simple birth and death process. Biometrika 43 (1956), 23–21.

The author considers a process with constant probabilities λdt and μdt of an individual giving birth or dying in $(t, t+dt)$. For a given $\tau>0$, let N_r=number of individuals existing at time $r\tau$, and consider the random variable $X_k=\sum_{r=1}^k N_r/\sum_{r=0}^{k-1} N_r$ as an estimator of $\alpha=\exp[(\lambda-\mu)\tau]$. It is shown that $E(X_k)<\alpha$ and $E(\log X_k)<\log\alpha$, although the bias is small if N_0 is large. The estimate is in general not consistent, in the sense that $\lim_{k\to\infty} E(X_k)=\alpha$ is not generally true, although it is at least true if $\mu=0$. When $\lambda=0$, the limiting bias is determined explicitly. The variance of X_k for large N_0 is determined (to order $1/N_0$) and is compared with the maximum likelihood estimator based on continuous observation on the interval $(0, t)$. For α in $(0.294, 16.1)$ and $k=1$, if $\tau\geqq 1.25t$ then X_1 has as low a variance as the maximum likelihood estimator. *T. E. Harris.*

Ramachandran, K. V. Contributions to simultaneous confidence interval estimation. Biometrics 12 (1956), 51–56.

The author shows how simultaneous confidence statements can be made about (a) the ratios of several variances by the use of the maximum F-ratio (s^2 max/s^2 min); (b) components of the main effects in factorial experiments by means of the studentized largest chi-square. *H. A. David* (Melbourne).

Scheffé, Henry. A "mixed model" for the analysis of variance. Ann. Math. Statist. 27 (1956), 23–36.

A "mixed model" is proposed in which the problem of the appropriate assumptions to make about the joint distribution of the random main effects and interactions is solved by letting this joint distribution follow from more basic and "natural" assumptions about the cell means. The expectations of the mean squares ordinarily calculated turn out, with suitable definition of the variance components, to have the same values as those usually found in more restrictive models, and some of the customary tests and confidence intervals are justified, but some aspects appear to be novel. For example the over-all test found for the fixed main effects and the associated muliple-comparison method require Hotelling's T^2. (Author's summary.) *D. M. Sandelius* (Göteborg).

Kudō, Tetsuo; Matsumura, Noboru; Dehara, Shigemi; Kōzai, Toshio; Sasaki, Kenichi; Umazume, Shigenori; and Watanabe, Yoshikatsu. Analyses of bimodal distributions. (On the decomposition of a bimodal distribution into two normal curves). J. Gakugei Tokushima Univ. Nat. Sci. Math. 6 (1955), 75–116.

Continuing research by Watanabe on bimodal distributions [same J. 5 (1954), 29–38; MR 16, 937] the present authors treat, independently of Karl Pearson's work, the problem of estimating by the method of moments the parameters of a mixture of two normal distributions. A large number of numerical illustrations are given. ω^2 and χ^2 tests are applied. In an addendum the authors compare their approach with that of K. Pearson. *D. M. Sandelius* (Göteborg).

Tiago de Oliveira, J. Distribution-free tests of goodness of fitting for distribution functions. Univ. Lisboa. Revista Fac. Ci. A. (2) 5 (1955–1956), 113–118.

Let $X_1, \cdots, X_n$ be the ordered values of n independent random variables, each distributed uniformly in (0, 1). The asymptotic distribution of $\sum_{i=1}^n (X_i-i(n+1)^{-1})^2$ is derived by a method of Anderson and Darling [Ann. Math. Statist. 23 (1952), 193–212; MR 14, 298]. *M. Dwass.*

Bose, R. C. Paired comparison designs for testing concordance between judges. Biometrika 43 (1956), 113–121.

It is required to compare n objects, by employing v judges. Each judge compares r pairs of objects ($r>1$). The configurations considered satisfy the following conditions: (a) Among the r pairs, compared by each judge, each object appears equally often, say α times. (b) Each pair is compared by k judges, $k>1$. (c) Given two judges, there are exactly λ pairs which are compared by both judges. These designs are called linked paired comparison designs (LPCD's).

A correspondence is set up between these designs and balanced incomplete block designs (BIBD's), by observing that each judge may be considered a treatment, and each pair a block. Thus, if a LPCD exists, then a BIBD exists. However, because of the restriction (a), the converse is not true.

Certain inequalities are proved, and then the construction problem is considered. It is shown how LPCD's with parameters $v=\frac{1}{2}(n-1)(n-2)$, $b=\frac{1}{2}n(n-1)$, $r=n$, $k=n-2$, $\lambda=2$, $\alpha=2$ can be constructed from BIBD's

with the same parameters (α is omitted), whenever such designs exist.

Next, it is shown how to construct LPCD's with parameters $n=2t$, $v=v^*$, $b=t(2t-1)$, $r=tr^*$, $k=k^*$, $\lambda=t\lambda^*$, $\alpha=r^*$ from any BIBD with parameters v^*, $b^*=2t-1$, r^*, k^*, λ^*. Finally, constructions are given for LPCD's with parameters $n=2t+1$, $v=v'$, $b=t(2t+1)$, $r=(2t+1)r'$, $k=k'$, $\lambda=(2t+1)\lambda'$, $\alpha=2r'$ by use of any BIBD with parameters v', $b'=t$, r', k', λ'. These last designs were found by M. G. Kendall [Biometrics **11** (1955), 43–62; MR **17**, 758].

W. S. Connor (Washington, D.C.).

Bradley, Ralph Allan. Rank analysis of incomplete block designs. III. Some large-sample results on estimation and power for a method of paired comparisons. Biometrika **42** (1955), 450–470.

The paper is an extension of earlier work [Biometrika **39** (1952), 324–345; **41** (1954), 502–537; MR **17**, 56, 57]. For the alternative hypothesis in which the t treatments have only two true ratings, it is shown that the test procedures reduce to those of the binomial or sign test.

For the alternative hypothesis in which all the true ratings $\Pi_1, \cdots, \Pi_t$ are different, an extensive investigation is undertaken. Let $p_1, \cdots, p_t$ denote the maximum likelihood estimators of $\Pi_1, \cdots, \Pi_t$, and let n denote the number of repetitions of the paired comparisons design. It is shown that $n^{\frac{1}{2}}(p_1-\Pi_1), \cdots, n^{\frac{1}{2}}(p_t-\Pi_t)$ have, for large values of n, the singular multivariate normal distribution of $(t-1)$ dimensions in a space of t dimensions with zero means and an appropriately defined dispersion matrix. Maximum likelihood estimators of the variances and covariances are given.

The limiting distribution of the test statistic T is found under the hypothesis H_a: $\Pi_i=t^{-1}+n^{-\frac{1}{2}}\delta_{in}$, where δ_{in} represents a sequence of constants converging to the constant δ_i as $n\to\infty$. The distribution is the non-central χ^2 distribution. This distribution is used to calculate the power of the T test, and to compare the power with that of other related tests.

In the author's model, the probability that treatment i will be preferred to treatment j is assumed to be $\Pi_i/(\Pi_i+\Pi_j)$. This model may be generalized by introducing a different probability Π_{ij} for every pair of treatments. The more general model implies the use of $t(t-1)/2$ independent binomial tests — which may be called a multi-binomial test. A less general model than the author's is that for which the analysis of variance is appropriate. For $t=3$ and 4 and for the .01 and .05 levels of significance, it is shown that asymptotically the power of T exceeds the power of the multi-binomial test, but that the power of T is less than the power of the analysis of variance F test. Asymptotic relative efficiencies are computed for $t=2, \cdots, 10, \infty$. *W. S. Connor.*

Welch, B. L. On linear combinations of several variances. J. Amer. Statist. Assoc. **51** (1956), 132–148.

The author considers a problem arising in the estimation of variance components, compound variances, etc.: the setting of confidence limits for a linear function (known coefficients) of several variances, the variances being individually estimated from independently distributed sums of squares. He discusses limits based on the normal and Type III approximations, and gives a method for improving the normal approximation.

P. Whittle.

Kudō, Hirokichi. On minimax invariant estimates of the transformation parameter. Nat. Sci. Rep. Ochanomizu Univ. **6** (1955), 31–73.

In a statistical decision problem which is invariant under a group G, there is a minimax solution invariant under G if three conditions are satisfied. Condition (1) is that G is locally compact and σ-compact and for every compact $J\subset G$ and every $\varepsilon>0$ there is a compact $K\subset G$ such that $\mu(K)>0$ and $\mu(K)/\mu(K\cdot J^{-1})>1-\varepsilon$, where μ is the right-invariant Haar measure on G. Condition (2) is too complicated to state here, but is satisfied if either the loss function is bounded or the set A of terminal actions is a topological space with respect to which the function $w(a)$, the loss from action a under a fixed distribution, is continuous and there is a sequence $\{C_n\}$ of compact subsets A such that $\bigcup C_n=A$ and $\inf_{a\notin C_n} w(a)\to\infty$ as $n\to\infty$. Condition (3) is also too complicated to state; it essentially requires that a certain set be measurable and that certain conditional probabilities behave in a regular way and is presumably usually satisfied. Minimax invariant procedures are described for the estimation of location and scale parameters, for the case in which G is a finite permutation group, and for the case in which G is the group of rotations in 3-space. *D. Blackwell.*

See also: Gheorghiu, p. 1095; Hammersley, p. 1136; Ludwig, p. 1095; Schützenberger, p. 1099.

Theory of Games, Mathematical Economics

Have

★ **Bellman, Richard. Dynamic programming and multistage decision processes of stochastic type.** Proceedings of the Second Symposium in Linear Programming, Washington, D.C., 1955, pp. 229–249. National Bureau of Standards, Washington, D.C., 1955.

The author briefly gives a slightly modified dynamic programming functional equation with a stochastic return function instead of the usual deterministic ones. [See Bellman, J. Operations Res. Soc. **2** (1954), 275–288; MR **15**, 975.] *T. L. Saaty* (Takoma Park, Md.).

Have

★ **Operations research for management. Vol. II. Case histories, methods, information handling.** Edited by Joseph F. McCloskey and John M. Coppinger. The Johns Hopkins Press, Baltimore, Md., 1956. xxxvi+563 pp. $8.00.

Vol. I appeared in 1954 [MR **16**, 501]. Papers of special mathematical interest are: The traveling-salesman problem (pp. 340–357) by M. M. Flood, and Bibliography of queueing theory (pp. 541–556) by Vera Riley.

Boldyreff, Alexander W. Determination of the maximal steady state flow of traffic through a railroad network. J. Operations Res. Soc. Amer. **3** (1955), 443–465.

The author gives an empirical method for reducing the two-directional flow problem of a complex railroad network to a undirectional flow. The following steady states conditions are enforced at three sets of points (origin, junction points and terminal points). 1. At the origins all trains leave and none arrive. 2. At the terminals all trains arrive and none leave. 3. At each junction point the number of trains coming in is equal to the number going out.

Then a brief discussion for the reduction of a network with many origins and terminals to an equivalent network with a single origin and a single terminal is given. The

following useful reductions are used. 1. The flow through two lines in series (parallel) cannot exceed the minimum (sum) of the capacities of the two lines. 2. Let two points P_1 and P_2 of the network be joined by a line of capacity C and S_1, S_2 are the sums of the capacities of all lines joined to P_1 and P_2 respectively. Then if

$$\min(S_1-C, S_2-C) \leqq C,$$

the line joining P_1 and P_2 may be dropped, allowing P_1 and P_2 to coincide.

The resulting network is equivalent to the original, with one less line and one less junction. This method is applied to a simple, but interesting model aimed at removing bottlenecks by organizing traffic through the network and thus maximizing its total flow.

T. L. Saaty (Takoma Park, Md.).

Whitin, T. M., and Youngs, J. W. T. A method for calculating optimal inventory levels and delivery time. Naval Res. Logist. Quart. 2 (1955), 157–173.

The paper considers (for a taxicab garage) the optimal stockage objective S which minimizes the expected loss $\phi(S, \mu)$ for carrying parts on shelves and for the loss due to idle time of the taxi's waiting for ordered parts. The demand for the parts is Poisson distributed with mean μ. It is found that

$$\phi(S, \mu) = S[C-(\Pi+C)P(S+1, \mu)] - \mu[C-(\Pi+C)P(S, \mu)],$$

where C is the cost of carrying a unit of stock for one day, Π the daily cost of being out of a part and

$$P(S, \mu) = \sum_{k=s}^{\infty} \frac{e^{-\mu}\mu^k}{k!}.$$

Then the optimal choice of S is the integer S_μ such that

$$P(S_\mu, \mu) \geqq \frac{C}{\Pi+C} > P(S_\mu+1, \mu).$$

T. L. Saaty (Takoma Park, Md.).

Simon, Herbert A. Dynamic programming under uncertainty with a quadratic criterion function. Econometrica 24 (1956), 74–81.

Let three finite random sequences of real numbers

$$P^{(t)} = (P_t, \cdots, P_{t+N})$$
$$S^{(t)} = (S_t, \cdots, S_{t+N})$$
$$I^{(t)} = (I_t, \cdots, I_{t+N})$$

satisfy the equations

$$(1) \qquad I_j = I_{j-1} + P_j - S_j \quad (j=t, \cdots, t+N).$$

Define "policy" as any sequence of functions

$$\phi^{(t)} = (\phi_t, \cdots, \phi_{t+N})$$

such that

$$P_j = \phi_j(0, S_t, S_{t+1}, \cdots, S_{j-1}) \quad (j=t, \cdots, t+N).$$

In particular, $P_t = \phi_t(0)$, a function with constant value. The policy $\phi^{(t)} = \hat{\phi}^{(t)}$ is called optimal if it minimizes the expectation of the quantity

$$C^{(t)} = \gamma(P^{(t)}, I^{(t)});$$

γ is called the "criterion function" and is supposed to be known. Write

$$\hat{P}_t = \hat{\phi}_t(0).$$

The following result is obtained. Let γ be quadratic with constant coefficients, to be denoted collectively by c. Then there exist $N+1$ numbers, $\lambda_k(c)$, $k=0, \cdots, N$, that depend on c in a known fashion and are such that

$$(2) \qquad \hat{P}_t = \sum_{k=0}^{n} \lambda_k(c) ES_{t+k},$$

where ES_j is the expectation of S_j.

Economic interpretation: During the jth month a firm produces P_j and sells S_j units; the inventory changes from I_{j-1} to I_j, according to (1). The firm decides about the amount to be produced, as follows: at the beginning of each month, say the tth, it plans a policy $\phi^{(t)}$ on the basis of an estimate of the distribution of $S^{(t)}$. The firm has to carry out only the first term of the sequence $\phi^{(t)}$, viz., $\phi_t(0) = P_t$. At the beginning of the $t+1$-th month the firm will estimate the distribution of $S^{(t+1)}$ and will determine (and carry out) P_{t+1}; and so on. The "criterion" $C^{(t)}$ is interpreted as the firm's total cost over the $N+1$ months beginning with t. Reasons are given why γ is (approximately) quadratic. If so, it is — by (2) — sufficient to know the coefficients c and the expectation (and no other distribution properties) of the sequence $S^{(t)}$ in order to compute optimal production, P_t, of the tth month.

Condition (1) was chosen for this particular illustration and can be replaced by a more general linear constraint without altering the result. When $N=0$ (as is the case in many other applications) the conditions can be relaxed still further [see H. Theil, Weltwirtschaftliches Arch. **72** (1954), 60–83].

J. Marschak (New Haven, Conn.).

Boiteux, M. Sur la gestion des monopoles publics astreints à l'équilibre budgétaire. Econometrica 24 (1956), 22–40.

Consider an economy composed of consumers, private producers (both operating under conditions of perfect competition), and public production units. Each of the latter is required to operate under a condition of budgetary equilibrium, in the sense that the net profits are fixed by law (for example, at zero). If y_i^h is the output of the ith commodity by the hth public firm (inputs being regarded as negative outputs) and p_i the price of the ith commodity, then the hth public firm must satisfy the condition $\sum_{i=1}^{n} p_i y_i^h = b_h$, for some b_h fixed from outside. What policy should the public sector adopt to achieve a Pareto optimum for the economy subject to the budgetary restraints just stated? The usual Lagrange multiplier technique for finding Pareto optima is used [see O. Lange, Econometrica **10** (1942), 215–228] except that prices and incomes of consumers are used as the variables instead of quantities. The nth commodity is taken as numeraire; it is assumed that consumers' incomes are paid in this commodity. Let (ij) be the market elasticity of substitution (sum of the elasticities of substitution for all consumers and private producers) between commodities i and j, z_i $(i=1, \cdots, n-1)$ defined as the solution of the equations, $\sum_{i=1}^{n-1} (ij) z_i = \sum_h \beta^h y_j^h$ $(j=1, \cdots, n-1)$, $t_i^h = z_i/(1+\beta^h)$, and $\pi_i^h = p_i - t_i^h$. Then the public firm h is to act as a perfect competitor with respect to the fictitious prices π_i^h. The deviations between fictitious and market prices have the interesting property that the ratio t_i^h/t_j^h is the same for all h. The parameters β^h have to be so

chosen that the resulting production policies satisfy the budgetary equilibrium conditions. *K. J. Arrow.*

Uzawa, Hirofumi. On the efficiency of Leontief's dynamic input-output system. Proc. Japan Acad. **32** (1956), 157–160.

A generalized discrete input-output system is defined essentially as a sequence $s^0, s^1, \cdots, s^t$ of nonnegative n-vectors satisfying

$$s^t=s^{t-1}+(I-A)x^{t-1} \text{ and } (B+A-I)x^t\leq s^{t-1}$$

where the x's are also nonnegative n-vectors and A and B are square matrices of nonnegative elements. An s-sequence is t-efficient if there is no possible sequence with the same initial vector whose terminal vector is componentwise bigger or equal. The author asserts that if the column-sums of A are less than one, B and $(C-I)$ are nonsingular and $C(C-I)^{-1}$ has nonnegative elements $(C=B(I-A)^{-1})$ then (1) a possible sequence s^0, s^1, s^2 is 2-efficient if and only if there are vectors x^0 and x^1 such that $s^1=s^0+(I-A)x^0$, $(B+A-I)x^0=s^0$,

$$s^1=s^2-(I-A)x^1,$$

$Bx^1=s^2$; and (2) a sequence is t-efficient if and only if each successive triplet is 2-efficient. No proofs are given, and indeed the "only if" part of (1) and the "if" part of (2) are false. *R. Solow* (Cambridge, Mass.).

Karlin, Samuel. The structure of dynamic programming models. Naval Res. Logist. Quart. **2** (1955), 285–294 (1956).

The theory of dynamic programming [cf. Bellman, Bull. Amer. Math. Soc. **60** (1954), 503–515; MR **16**, 732] introduces a class of functional equations of the form $f(p)=\max_q G(f(T(p, q)), q)$, in connection with the study of multistage decision processes.

In this paper, the author considers both deterministic and nondeterministic processes and demonstrates that, under certain assumptions, the existence of an optimal policy may be deduced from Tychonoff's theorem concerning the direct product of compact spaces. This follows the method given in an earlier unpublished paper by H. N. Shapiro and the author.

Following this, there is a derivation of the "principle of optimality", cf. the article cited above, for these processes, which yields the fundamental functional equations. Existence and uniqueness theorems are derived for some particular equations, including the noteworthy "optimal inventory" equation of Arrow, Harris and Marschak [Econometrica **19** (1951), 250–272; MR **13**, 368]. The results are to a large extent similar to those presented in Bellman, Trans. Amer. Math. Soc. **80** (1955), 51–71 [MR **17**, 632] but the methods are quite different.

R. Bellman (Santa Monica, Calif.).

Siegel, Sidney. A method for obtaining an ordered metric scale. Psychometrika **21** (1956), 207–216.

Implications of the von Neumann-Morgenstern scheme of utility measurement are explored in the context of a small number of ultimate outcomes and of simple dichomous bets with equiprobable outcomes. This situation of course leads to rather less than a fully numerical scale. The redundancies that give the theory content are exposed. A little experimental work is reported.

L. J. Savage (Chicago, Ill.).

Luce, R. Duncan; and Adams, Ernest W. The determination of subjective characteristic functions in games with misperceived payoff functions. Econometrica **24** (1956), 158–171.

The authors propose a modification of the theory of n-person games in which the assumption that each player correctly perceives the payoff functions for the other players is dropped. It is replaced by the assumption that each player has his own perception of the payoff function for each of the other players. There are then n "subjective" characteristic functions, one for each player. It is shown that these subjective characteristic functions are determined in the context of reasonable assumptions by the preferences of the players for the various coalitions. The misperceived payoff functions determine the forces on the realignment of coalitions as well as their original formation. An outline for an equilibrium theory of the resulting game theory is indicated. The resulting stability concept is a generalization of the concept of ψ-stability previously introduced by Luce Mathematical models of human behavior, Dunlap, Stamford, Conn., 1955, pp. 32–44. *E. Nering* (Minneapolis, Minn.).

Isaacs, Rufus. The problem of aiming and evasion. Naval Res. Logist. Quart. **2** (1955), 47–67.

The paper is a discussion of problems involving the maneuvering of an evader E (a ship) to avoid destruction by a marksman P (an aircraft) with time lag between detection and the hit of the weapon in P's estimation of E's position. When considered as a game, E's best strategy and value of the game are given by L. E. Dubins, [Inst. Air Weapons Res. Tech. Note no. 2] and by R. Isaacs and S. Karlin [Rand Res. Memo. no. RM-1316 (1954)]. The difficult problem of finding P's strategies is studied here. P has no optimal strategy. He is found to have an ideal strategy with the property that every near optimal (ε strategy) is close to it. A near optimal strategy is defined as a strategy which assures P of a hit with probability $\geq V-\varepsilon$ for $\varepsilon>0$ and no strategy insures V, where V is the value of the game to E. P also has a class of passive strategies such that if and only if he obeys their dictates will he either come within ε of the best hit probability or else always remain in a position where it is possible to do so.

T. L. Saaty (Takoma Park, Md.).

Kose, T. Solutions of saddle value problems by differential equations. Econometrica **24** (1956), 59–70.

The author considers the problem of finding the saddle point of a function $\phi(x, u)$ of two vectors by methods analogous to the gradient method for finding the maximum of a function of one variable. The following theorem is asserted: If ϕ is differentiable in both variables, strictly concave in x and strictly convex in u, then all solutions of the following system of differential equations converge to a unique saddle point: $\dot{x}_i=\delta_{x_i}\phi_{x_i}$, $\dot{u}_j=-\delta_{u_j}\phi_{u_j}$, where $\delta_{x_i}=0$ if $x_i=0$ and $\phi_{x_i}>0$ and 1 otherwise, $\delta_{u_j}=0$ if $u_j=0$ and $\phi_{u_j}>0$ and 1 otherwise. (The theorem actually stated is very slightly more general than that given here.) The factors δ_{x_i}, δ_{u_j} serve to keep the variables x, u nonnegative. However, as the author acknowledges in a footnote, his proof is not rigorous (however, the theorem is true).

He then considers the case where ϕ is linear in both x and u; in this case, the above method eventually oscillates. He presents an alternative system of differential equations, where ϕ_{x_i} is replaced everywhere in the above system by $\phi_{x_i}+\gamma\phi_{x_i t}$ and ϕ_{u_j} by $\phi_{u_j}+\gamma\phi_{u_j t}$, $\gamma>0$. He

asserts again that this system converges to the saddle-point, but the proof does not appear rigorous. The paper concludes with brief discussions of analogous devices for solving these systems and of applications to the dynamic Leontief model. *K. J. Arrow* (Stanford, Calif.).

de Lasala, Jesus. Contribution to the study of a quick method of calculation for application of the simplex method to Hitchcock's transportation problem. Trabajos Estadist. 6 (1955), 209–236. (Spanish. English summary)

The author considers the solution of the Hitchcock transportation problem by a method based on the arrangement in a tableau of the column-vectors P_{ij}, corresponding to the requirement equations, [G. B. Dantzig, Activity Analysis of Production and Allocation, Wiley, New York, 1951; MR 15, 48; and B. Giardina, A. Longo and S. Ricossa, La programmazione lineare nell'industria, Unione Industriale di Torino, 1954]. Special attention is given to the case of the distribution of empty cars in a railroad system. The relations of linear dependence among the vectors P_{ij} are studied according to their relative positions in the tableau. The conclusions obtained are used in order to simplify the selection of the initial basis and the procedure of changing basis. Some indications for dealing with degeneracy and others difficulties are given and a example is thoroughly examined. *A. G. Azpeitia* (Providence, R.I.).

Freeman, Raoul J. Linear programming. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. **955** (1955), 25 pp.

The linear programming problem and its solution by the Simplex Method are presented with a numerical example. Several other methods are briefly surveyed and their relations with machine computation are discussed. The use of Ordvac is specially considered. The transportation problem is sketched and a flow chart for its solution is appended. *A. G. Azpeitia.*

See also: Frame, p. 1136; Varoli, p. 1136.

Mathematical Biology

Huron, R. Sur l'interprétation mathématique des groupages sanguins. Méthode et tableaux d'utilisation. Ann. Fac. Sci. Univ. Toulouse (4) **19** (1955), 1–116 (1956).

This paper gives tables for the calculation of certain probabilities in the genetics of blood groups. The probabilities considered include (among other less important ones) the probabilities of the various possible gene combinations for an individual given his blood group, and the probability that a child has a given blood group, when that of the mother is known, or when the blood groups of both parents are known. The blood groups considered are A_1A_2BO, Rhesus (tested for C, c, D, E only), and MN (not tested for S or s). The tables could be applied to cases of disputed paternity, where the alleged father is not completely excluded by the blood group tests and it is desired to know whether the accusation is plausible or not. The author proposes as an index of plausibility the probability that the real father should have the same blood group as the alleged father, when the blood groups of mother and child are taken as fixed. He shows how this can be found from the tables, and gives some numerical examples. *C. A. B. Smith* (London).

Komatu, Yusaku; and Nishimiya, Han. Probabilities on inheritance in consanguineous families. I, II, III, IV, V, VI, VII, VIII, IX, X, XI, XII, XIII. Proc. Japan Acad. **30** (1954), 42–45, 46–48, 49–52, 148–151, 152–155, 236–240, 241–244, 245–247, 636–640, 641–649, 650–654; **31** (1955), 186–189, 190–194.

The authors give a remarkable wealth of formulae relating to the joint probability distribution of the genotypes of relatives with respect to an allelomorphic series $A_1, A_2, \cdots, A_m$ in an infinite population which is assumed to be in equilibrium under random mating except for certain specified consanguineous marriages. Various systems of continued inbreeding are examined indetail.

Mathematical proofs of the basic formulae appear in the paper reviewed second below. *A. R. G. Owen.*

Komatu, Yûsaku. Probabilistic investigations of population genetics. Kôdai Math. Sem. Rep. **7** (1955), 1–7.

This paper considers the joint probability distribution of the genotypes of relatives with respect to an allelomorphic series $A_1, A_2, \cdots, A_m$ in an infinite population which is in equilibrium under random mating except for certain specified consanguineous marriages. The author sketches the mathematical methods which have been used to derive the formulae given in the paper reviewed above. *A. R. G. Owen* (Cambridge, England).

Komatu, Yûsaku; and Nishimiya, Han. Probabilistic investigations on inheritance in consanguineous families. Bull. Tokyo Inst. Tech. Ser. B. **1954**, 1–66, 67–152, 153–222; **1955**, 65–113.

On the assumption of the Mendelian segregation of alleles $A_1, A_2, \cdots, A_m$ in an effectively infinite population in equilibrium under random mating, the authors determine the simultaneous probability distribution of the genotypes of various related pairs of individuals. Previous papers have dealt with the cases of parent and children, and of siblings [Proc. Japan. Acad. **27** (1951), 587–620, 689–699; **39** (1953), 68–82; MR **17**, 59]. The present paper gives formulae for the probability of each of the following events: (i) that an individual be $AaAb$ and its nth lineal descendant be A_cA_d, (ii) that two individuals be A_aA_b, A_cA_d and their nth lineal descendant be A_eA_f, (iii) that one of two individuals be A_aA_b and that one of their kth descendants be A_cA_d and that one of their lth descendants be A_eA_f, (iv) that a kth and an lth descendant of the same couple be respectively A_aA_b and A_cA_d. Formulae are also derived appropriate to a cousinship of any degree of removal. More complex relationships are also treated; for example, the probability that two individuals be A_aA_b and A_cA_d and the nth descendant of a marriage between one of their kth descendants and one of their lth descendants be A_eA_f. By further generalization results are got for cases of very close inbreeding. *A. R. G. Owen.*

Peyovitch, T. Application de mathématique à la biologie. Bull. Soc. Math. Phys. Serbie **6** (1954), 199–208. (Serbo-Croatian. French summary)

Mandelbrot, Benoit. La distribution de Willis-Yule, relative aux nombres d'espèces dans les genres biologiques. C. R. Acad. Sci. Paris **242** (1956), 2223–2226.

Rashevsky, N. Contributions to topological biology: some considerations on the primordial graph and on some possible transformations. Bull. Math. Biophys. **18** (1956), 113–128.

TOPOLOGICAL ALGEBRAIC STRUCTURES

Topological Groups

de Groot, J. Orthogonal isomorphic representations of free groups. Canad. J. Math. 8 (1956), 256–262.

L'auteur précise des théorèmes connus sur la possibilité de plonger un groupe libre de rang 2 (ou de rang infini ayant la puissance du continu) dans le groupe réel orthogonal à 3 variables, en donnant des représentations explicites des générateurs de ces groupes. La remarque 4.2 de la p. 261 est obscure: l'auteur semble vouloir dire que tout groupe libre est sousgroupe d'un groupe orthogonal, ce qui est visiblement inexact, la puissance d'un groupe orthogonal étant la puissance du continu. *J. Dieudonné.*

Plotkin, B. I. On groups with finiteness conditions for abelian subgroups. Dokl. Akad. Nauk SSSR (N.S.) 107 (1956), 648–651. (Russian)

A group is called a WF-group provided it has a WF-series, that is, an ascending normal series every factor of which is finite or locally nilpotent. Necessary and sufficient that a group G be a WF-group is that G be the extension of a radical group by a semi-simple WF-group. If every abelian subgroup in a WF-group is either finite or satisfies the maximal condition or satisfies the minimal condition, then the group itself has, respectively, the same property. If every abelian subgroup of a nilgroup G has type A_4 in the sense of Mal'cev [same Dokl. (N.S.) 67 (1949), 23–25; MR 11, 78], then G is a nilpotent A_4-group. *R. A. Good* (College Park, Md.).

Kimura, Naoki; and Tamura, Takayuki. Compact mob with a unique left unit. Math. J. Okayama Univ. 5 (1956), 115–119.

After establishing some properties of a compact semigroup with a unique left unit which is not a right unit the authors characterize such a semigroup in terms of a class of functions taking the semigroup into itself. *A. D. Wallace* (New Orleans, La.).

Ghika, Al. Ensembles entiers, convexes, serrés et absorbants, dans les groupes à radicaux. Com. Acad. R. P. Romîne 5 (1955), 1229–1233. (Romanian. Russian and French summaries)

Ghika, Al. Groupes topologiques localement paraconvexes. Com. Acad. R. P. Romîne 5 (1955), 1235–1240. (Romanian. Russian and French summaries)

The author's aim is to extend to certain topological groups the existing theory of convexity in topological linear spaces. These two notes supply only definitions and statements of theorems; proofs will appear elsewhere. A subset A of a group G is called "convex" if for all natural numbers m and n, $x^m \in A^m$ implies $x \in A$, and $A^{(m/n)} = (A^{(1/n)})^m$, where $A^{(m/n)} = \{x^m : x^n \in A\}$; A is "paraconvex" provided xA is convex for some $x \in G$. For a topological group with radical, the notion of local paraconvexity is introduced, and analogues of some of the usual theorems stated (generation of the topology by seminorms, etc.). *V. L. Klee, Jr.* (Los Angeles, Calif.).

Goetz, A. Über eine hinreichende Bedingung für die Existenz einer invarianten Metrik in homogenen Räumen. Bull. Acad. Polon. Sci. Cl. III. 3 (1955), 467–469.

Let G be a topological group with left invariant metric ϱ, let H be a closed subgroup, and let $d(xH, yH) = \inf\{\varrho(u, v) : u \in xH \text{ and } v \in yH\}$. Then d, which is clearly invariant under the operations by members of G on the homogeneous space G/H, is a metric for G/H provided either ϱ is invariant under right translation by members of H, or H is a normal subgroup. It may be remarked that in either of these cases d is identical with the Hausdorff metric for G/H. *J. L. Kelley* (Berkeley, Calif.).

Borel, Armand. Sur la torsion des groupes de Lie. J. Math. Pures Appl. (9) 35 (1956), 127–139.

The principal results of this paper are: (Theorem I) If G is a compact connected Lie group and p is a prime which does not divide the order of the Weyl-group of G, then G has no p-torsion .(Theorem II) If p does not divide any of the coefficients of the dominant root of G, then the classifying space of G has no p-torsion. These general theorems yield explicit new results only in the cases E_6, E_7, E_8, because the torsion of the other simple groups has already been determined by the author. Thus from Theorem I it follows that E_6, E_7, and E_8 have no p-torsion for $p \geqq 7$, $p \geqq 11$, $p \geqq 11$ respectively, while the second result implies that their classifying spaces have no p-torsion for $p \geqq 5$, $p \geqq 5$, $p \geqq 7$ respectively.

The proof of Theorem I is general in nature while Theorem II is proved by verification. The author makes repeated use of the following lemma: If in a differentiable fiber space, E, p does not divide the Euler number of the fiber, then the cohomology mod p of the base is injected isomorphically by the projection into the cohomology of E. *R. Bott* (Princeton, N.J.).

Schwarz, Štefan. Remark on the theory of bicompact semigroups. Mat.-Fyz. Časopis Slovensk. Akad. Vied 5 (1955), 86–89. (Slovak. Russian summary)

Let S be a compact (Hausdorff) semigroup such that $ax = ay$ implies $x = y$. Then S is the union of disjoint isomorphic groups, and S contains no proper left ideals. The known result that S is a group if both cancellation laws hold is also proved. [For a stronger result concerning arbitrary semigroups in which every element has finite order, see Hewitt and Zuckerman, Acta Math. 93 (1955), 67–119, Th. 2.12; MR 17, 1048.] *E. Hewitt.*

Wright, Fred B. Semigroups in compact groups. Proc. Amer. Math. Soc. 7 (1956), 309–311.

The author first proves the following theorems. Theorem I: Let K be a compact group and S a semigroup in K. Then S is necessarily a closed subgroup under either of the following conditions: (1) S is closed, (2) S is open. Theorem II: Let K be a compact group, and S a semigroup which is of the second category at the identity of K and which satisfies the condition of Baire. Then S is a subgroup of K which is both open and closed. Theorem III: In a compact group K any locally compact subsemigroup S is actually an open and closed subgroup.

An angular semigroup in an abelian topological group is an open semigroup having the identity element as a limit point. Hille and Zorn [Ann. of Math. (2) 44 (1943), 554–561; see also Hille, Functional analysis and semigroups, Amer. Math. Soc. Colloq. Publ., v. 31, New York, 1948, chap. VII; MR 5, 40; 9, 594] have classified the anuglar semigroups of additive Euclidean n-space. The author, using Theorem I and the structure theory for

abelian groups, determines the angular semigroups in any locally compact abelian group. *A. Shields.*

Krishnan, V. S. Additivity and symmetry for generalised uniform structures, and characterisations of semi-uniform structures. J. Madras Univ. Sect. B. **25** (1955), 201–212.

By weakening or omitting various of the postulates for spaces with uniform structure, the author obtains and studies generalizations called generalized uniform structures. If such a space satisfies a weak transitivity law it is called a semi-uniform structure. The semi-uniform structures, with or without conditions of additivity or symmetry, are characterized in terms of particular structures determined by distance functions called semi-écarts, which satisfy only the triangle law for metric spaces. A sample theorem is that any semi-uniform structure is the lattice product of a class of comparable semi-écart structures.

As an application, the author proves an imbedding theorem to the effect that any semi-uniform totally ordered commutative semi-group with unit element and cancellation laws can be imbedded in a symmetric semi-uniform ordered group. This generalizes the imbedding of the ordered semi-group of nonnegative real or rational numbers in the ordered group of reals or rationals.

O. Frink (University Park, Pa.).

See also: Fort, p. 1115; Nagahara and Tominaga, p. 1046.

Lie Groups, Lie Algebras

Tôgô, Shigeaki. On the derivations of Lie algebras. J. Sci. Hiroshima Univ. Ser. A. **19** (1955), 71–77.

The author studies the relations between the derivation algebras $D(L)$ and $D(R)$, where L is a finite dimensional Lie algebra over a field of characteristic O and R is the radical of L. He shows first that a derivation of L is an inner derivation whenever its restriction to R coincides with the restriction to R of an inner derivation of L. The next main result strengthens a result of G. F. Leger [Proc. Amer. Math. Soc. **4** (1953), 511–514; MR **15**, 6]. If M is a Lie algebra, and N is an ideal of M, one says that M splits over N if the Lie algebra extension $N\to M\to M/N$ is split; $I(L)$ denotes the ideal of $D(L)$ consisting of the inner derivations of L; the result is that the following three conditions are equivalent: (1): $D(L)$ splits over $I(L)$; (2): the natural image of $D(L)$ in $D(R)$ splits over the natural image of $I(L)$ in $D(R)$; (3): the natural image of $D(L)$ in $D(R)$ splits over $I(R)$. Finally, the author generalizes a well-known result on the derivation algebra of a semisimple Lie algebra to reductive Lie algebras. The critical new result is as follows: $D(L)$ is completely reducible (i.e., L is semisimple as a $D(L)$-module) if and only if the center Z of L is at most 1-dimensional and L is the direct sum of Z and a semisimple ideal; in that case, $D(L)$ is isomorphic with L. *G. Hochschild.*

Seligman, George B. On Lie algebras of prime characteristic. Mem. Amer. Math. Soc. no. **19** (1956), 85 pp.

Let L be a restricted Lie algebra [see N. Jacobson, Trans. Amer. Math. Soc. **50** (1941), 15–25; MR **3**, 103] over an algebraically closed field F of characteristic $p>7$. Assume that i) L has a restricted representation $x\to U(x)$ such that the symmetric bilinear form

$$(x, y)=\operatorname{trace} U(x)U(y)$$

is non-degenerate; and ii), L contains no non-zero abelian ideals. Any Lie algebra over F whose Killing form is non-degenerate may be viewed as a restricted Lie algebra satisfying i) and ii). The algebra L is a direct sum of simple restricted ideals L_i upon each of which the restriction of the form (x, y) is non-degenerate, so that the structure of L is entirely known when the structure of the simple algebras L_i is given. In this paper the simple Lie algebras L satisfying the hypotheses listed above are completely determined; they turn out to be the precise analogous of the simple Lie algebras over the field of complex numbers.

At many points the discussion is reminiscent of the classical procedure for determining the simple complex Lie algebras. The techniques of proof used in the characteristic zero theory, however, are largely inapplicable, and are replaced by new arguments due to Zassenhaus, Jacobson, and the author. Let H be a fixed Cartan subalgebra of L. Then H is abelian, and the restriction of the form (x, y) to H is non-degenerate. The roots $\alpha, \beta, \cdots$ of L with respect to H are introduced, and it is proved that the spaces $L_\alpha=\{u_i|[uh]=\alpha(h)u, h\in H\}$ corresponding to the roots are one dimensional. For each $\alpha\neq 0$, there exists a unique element $h_\alpha\in H$ such that $(h_\alpha, h)=\alpha(h)$ for all $h\in H$. A system of roots $\alpha_1, \cdots, \alpha_k$ is simple if $\alpha_i-\alpha_j$ is not a root, $1\leq i, j\leq k$, and indecomposable if for each i there exists a $j\neq i$ such that $\alpha_i(h_{\alpha_j})\neq 0$. All possible indecomposable simple systems are explicitly determined. It is then shown, by calculating all possible roots which can be linearly dependent upon the roots belonging to a given indecomposable simple system, that L possess an indecomposable simple system, called a maximal simple system $\alpha_1, \cdots, \alpha_l$ such that $h_{\alpha 1}, \cdots, h_{\alpha l}$ form a basis for H, and such that $\alpha_1, \cdots, \alpha_l$ generate, in a certain sense, all roots of L relative to H. Two simple algebras which possess maximal simple systems of the same class are shown to be isomorphic. The main result states that L must either belong to one of four infinite classes of simple algebras, corresponding, with a single possible exception in the first class which doesn't appear in the charactriestic zero theory, to E. Cartan's classes A, B, C, or D, or L must be one of five exceptional algebras, which are the analogues of the exceptional algebras G_2, F_4, E_6, E_7 and E_8. The author then exhibits examples of algebras belonging to each of the classes he has determined.

C. W. Curtis (Los Angeles, Calif.).

Morinaga, Katutaro; and Nôno, Takayuki. On the matrix space. J. Sci. Hiroshima Univ. Ser. A. **19** (1955), 51–69.

The authors prove that the set of all regular complex matrices of degree n which have no real negative characteristic root is a maximal simply connected domain in the full linear group, and a similar statement for the complex orthogonal group. They derive various properties of the arcs of one-parameter groups in $SL(n, C)$ or $SO(n, C)$ joining the unit element to any given element.

C. Chevalley (Paris).

Borel, Armand. Kählerian coset spaces of semisimple Lie groups. Proc. Nat. Acad. Sci. U. S. A. **40** (1954), 1147–1151.

L'A. étudie, dans cette note, les espaces homogènes G/H où G effectif est semi-simple et qui admettent une

structure kählerienne invariante par G. Les principaux résultats sont les suivants: si G/H (G semi-simple) est kählerien homogène, H est compact, connexe et égal au centralisateur d'un tore de G; le centre de G est réduit à $\{e\}$ et G/H est simplement connexe. Par produit on peut se ramener au cas où G est simple. Les conclusions sont les mêmes si G est semi-simple compact et $V_{2n}=G/H$ homogène symplectique (c'est-à-dire admet une 2-forme fermée de rang $2n$ invariante par G).

Inversement si G est semi-simple, compact, et H le centralisateur d'un tore, G/H est homogène kählerien algébrique, et admet une décomposition cellulaire analytique complexe en cellules ouverts qui sont birationellement et birégulièrement équivalentes à des espaces affines complexes; en particulier l'homologie entière est sans torsion.

Si G est simple, non compact, de centre réduit à $\{e\}$, soit K un sous-groupe compact maximal de G, H un sous-groupe de K centralisateur d'un tore dans G; G/H est homogène complexe et homogène symplectique. Pour qu'il soit homogène kählerien, il faut et il suffit que G/K soit hermitien symétrique et la fibration de G/H par K/H est analytique complexe.

Par produit, on construit alors immédiatement les espaces homogènes kählериens à G semi-simple. Un domaine borné homogène de C^n étant un espace homogène kählerien qui n'admet aucune sous-variété analytique compacte de dimension non nulle, il en résulte que si un tel domaine admet un groupe transitif semisimple d'automorphismes, il est symétrique.

A. Lichnerowicz (Paris).

Koszul, J. L. Sur la forme hermitienne canonique des espaces homogènes complexes. Canad. J. Math. **7** (1955), 562–576.

Let G be a connected Lie group and B a closed subgroup of G. Suppose that the homogeneous space G/H has a complex structure I and a volume element ω, both invariant under G. This ω is unique up to a constant multiple. Locally, it can be written as

$$\omega=K dz_1\wedge\cdots\wedge dz_n\wedge d\bar z_1\wedge\cdots\wedge d\bar z_n,$$

where $z_1, \cdots, z_n$ are the local complex coordinates of G/B. The tensor field $\partial^2 \log K/\partial z_i\partial\bar z_j$ will be called the canonical hermitian form of G/B and denoted by h. This form h depends on the pair (G, B) as well as the invariant complex structure I of G/B. The author first obtains a formula to calculate h, and then proves, among other results, the following: (i) If G is semi-simple and G/B has a non-degenerate canonical hermitian form, then B is an open subgroup of the centralizer of a one-parameter subgroup of G; (ii) let G be semi-simple, and L be a one-parameter subgroup of G. If B is compact and is an open subgroup of the centralizer of L, then B must be connected and the homogeneous space G/B has invariant complex structures. For each of these invariant complex structures, the canonical hermitian form h is non-degenerate. The number of negative squares in h is equal to the difference of the dimension of a maximal compact subgroup of G and the dimension of B; (iii) if G is compact, any non-degenerate canonical hermitian form is negative definite.

From (i), (ii) and some well known results, it follows immediately that if the group of analytic automorphisms of a homogeneous bounded domain D is semi-simple, then D must be symmetric in the sense of E. Cartan. A. Borel has also proved this result in the paper reviewed above.

H. C. Wang (Seattle, Wash.).

Tits, J. Sur les groupes doublement transitifs continus: correction et compléments. Comment. Math. Helv. **30** (1956), 234–240.

L'auteur remarque que dans son énumération de groupes doublement transitifs [mêmes Comment. **26** (1952), 203–224; MR **14**, 447] manque le groupe des $x\to a\times b+c$, où x, a, b, c tous des quaternions, $|a|=1$, et où b parcourt un groupe multiplicatif qui ne fait pas partie de celui des quaternions de module 1. L'énumération de presque-corps doit être complété dans un sens analogue.

H. Freudenthal (Utrecht).

Berezin, F. A. Laplace operators on semisimple Lie groups. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 9–12. (Russian)

Let G be a Lie group which acts transitively on the manifold M. A Laplace operator on M (with respect to G) is a differential operator which commutes with all "translations" by members of G. Let x_0 be a point of M and let G_0 be the subgroup of all $g\in G$ such that $g(x_0)=x_0$. The radial part of the Laplace operator Δ is defined to be the restriction of Δ to the set of those functions in its domain which are invariant under the action of G_0. The author first describes the most general operator which is the radial part of some Laplace operator in each of the following two cases: A) $G=H\times H$, $M=H$, H is semi-simple and $(g_1, g_2)(x)=g_1^{-1}xg_2$. B) $M=G/G_n$, G is complex and semisimple, G_n is a maximal compact subgroup and the action of G on M is the canonical one. Then he shows how the results may be applied to obtain certain results of Gel'fand, Neumark, and Harish-Chandra about the representations and characters of semisimple Lie groups.

G. W. Mackey (Cambridge, Mass.).

See also: Cartan and Eilenberg, p. 1040; Kostant, p. 1128; Witt, p. 1050.

Topological Vector Spaces

Have

★**Bourbaki, N. Eléments de mathématique. XVIII. Première partie: Les structures fondamentales de l'analyse. Livre V: Espaces vectoriels topologiques. Chapitre III: Espaces d'applications linéaires continues. Chapitre IV: La dualité dans les espaces vectoriels topologiques. Chapitre V: Espaces hilbertiens.** Actualités Sci. Ind., no. 1229. Hermann & Cie, Paris, 1955. ii+191 pp. 2000 francs.

Have

★**Bourbaki, N. Eléments de mathématique. XIX. Première partie: Les structures fondamentales de l'analyse. Livre V: Espaces vectoriels topologiques. (Fascicule de résultats.)** Actualités Sci. Ind., no. 1230. Hermann & Cie, Paris, 1955. ii+39 pp. 400 francs.

These volumes complete Book V of the Bourbaki program, and permit an overall evaluation of the treatment of topological vector spaces. Book V is unquestionably an excellent text, informative, highly organized and disciplined, with (almost everywhere) brutal suppression of all but essential facts. The authors have exercised remarkable judgement in avoiding the proliferation of concept and definition which is common in a subject as new as that of topological vector spaces. My chief criticisms of this second part of the book are these: There are several results (in particular the Eberlein-Krein-Šmulian-Grothendieck work on weak compactness) which deserve, but do not get, a full exposition in the text. Further, the

problems, which contain many beautiful applications and extensions of the theorems of the text, are neither titled nor indexed, and so are practically useless for reference purposes. Finally, it may be mentioned that the last chapter and the last section of the preceding chapter are largely concerned with metric invariants, and as such are a digression from the stated subject matter. A brief description of the material follows.

The presentation in chapter III (Space of continuous linear functions) is centered around the notion of a tonnelé space (a locally convex space in which each disk, that is, each closed circled convex set which absorbs points, is a neighborhood of 0). After a preliminary discussion of bounded sets, products, and inductive limits, the space $\mathcal{L}(E, F)$ of continuous linear mappings of E into F is investigated. This space is topologized by uniform convergence on the members of a family $\mathcal{S}$ of bounded subsets of E– each of the standard topologies corresponds in this fashion to a family $\mathcal{S}$. Bounded subsets of $\mathcal{L}(E, F)$ and equicontinuity are investigated, the major theorem being that a simply bounded (i.e. relative to the topology of pointwise convergence) subset of $\mathcal{L}(E, F)$ is equicontinuous whenever E is tonnelé and F locally convex. A theorem deducing continuity on a product from separate continuity is established, and a completeness theorem for subsets of $\mathcal{L}(E, F)$ is proved. The concept of hypocontinuous bilinear function is investigated (this weakened form of continuity is the natural one for topological vector spaces; continuous bilinear functions occur primarily in the normed space case).

Chapter IV, Duality in topological vector spaces, begins with a discussion of "spaces in duality" (two vector spaces and a bilinear functional on the product). The weak topologies are defined, the weakly continuous linear functionals and the weak topologies for subspaces, quotients, and products are identified. (An innovation: the polar of a set M is all y such that the real part of $\langle x, y\rangle \leq 1$ for all x in M. This yields stronger results than the usual absolute value definition.) The "faible" (=weak) and "affaiblie" (=weak*) topologies for a locally convex space and its dual are defined, and the property of being tonnelé is characterized in terms of the dual space. The class of topologies yielding a given dual is described (Mackey theorem), and it is shown that a convex subset of the dual of a Fréchet space is weak* closed if its intersection with each equicontinuous set A is weak* closed in A. The duals of subspaces, products etc. are identified. Elementary properties of the strong dual are listed, semi-reflexive (evaluation map into the second adjoint is onto) and reflexive (evaluation is a topological isomorphism) spaces are characterized. Montel spaces (abstraction of properties possessed by certain spaces of analytic functions) are discussed briefly. The standard continuity and openness theorems for a linear transformation and its adjoint are established. The chapter concludes with a section on duality in Banach spaces.

The last chapter is devoted to the elementary theory of Hilbert spaces, including the identification of the dual, direct sums, and orthonormal bases.

In conclusion, I would like to make three minor comments. The theorem credited to Mackey in the text and in this review was first proved by Šmulian [Mat. Sb. N.S. 7(49) (1940), 425–448; MR 2, 102]. The definition of a linear topological space is due to A. Kolmogoroff [Studia Math. 5 (1935), 29–33]. Lastly, the Banach-Steinhaus theorem amounts to the two following geomitric statements: each tonneau in a Banach space is a neighborhood of 0, and the intersection of tonneaus is a tonneau provided this intersection absorbs points. The proposition labelled "Banach-Steinhaus" in the text is a consequence of the second of these statements only, and so gives a slightly distorted view of the content of the original theorem.
J. L. Kelley (Berkeley, Calif.).

★ **Grothendieck, A. Espaces vectoriels topologiques.** Instituto de Matemática Pura e Aplicada, Universidade de São Paulo, São Paulo, 1954. 240 pp. (miméographiées)

This text is taken from lectures delivered at São Paulo by the author; it is intended to combine these lectures with those of a similar course given by L. Nachbin in a more formal presentation — a consummation devoutly to be wished. Naturally, this volume is less polished and less highly organized than the Bourbaki text; there is an exhausting (but not exhaustive) list of typographical errors and omissions, and no index is given. However, this book covers the material very well (including a great deal which I could not find in the Bourbaki text), the proofs are (on the whole) elegant and informative, and a number of applications of the theory to particular spaces makes for interesting reading. In brief, this is a substantial contribution to the literature, and will be welcomed in spite of its shortcomings as to form. An indication of the contents follows.

Chapter 0 is a topological introduction, devoted to projective limits, function spaces, and other topics of especial interest to the theory of topological vector spaces. Chapter 1, on general properties of topological vector spaces (over the real or complex numbers) deals with elementary properties, products, subspaces, quotients, spaces of continuous linear functions, semi-norms, normability, uniqueness of topology for finite dimensional separated spaces, the precompact neighborhood characterization of finite dimensionality, the closed graph theorem, the Banach-Steinhaus theorem (approximately the classical form), and (after Bourbaki) a theorem on joint continuity as a consequence of separate continuity. A number of excellent exercises are interspersed, here and throughout the text.

Chapter 2, on duality, contains the relation between convex cones and order, semi-norms, the Hahn-Banach theorem, dual systems (spaces in duality), weak topologies, weakly continuous functionals, polars (as in Bourbaki), the Mackay theorem as quoted above, biduals (topologized by uniform convergence on equicontinuous sets; not the dual of the dual), reflexive spaces (=semi-reflexive in the sense of Bourbaki), completion of a locally convex space (a topic slighted by Bourbaki; one of the few inefficient proofs I noticed in this text), subspaces, quotients, products, projective limits, tensor products, transposes (adjoints), continuity and openness, reflexive Banach spaces, elementary properties of weak compactness, and Montel spaces (not the Bourbaki definition).

Chapter 3 is devoted to spaces of linear mappings, the topology of uniform convergence on a family $\mathcal{S}$ of sets, bounded sets, equicontinuity, tonnelé spaces, bornological spaces, continuous and hypocontinuous bilinear functions, a space of continuous functions $(C(M, E) \approx C(M) \mathbin{\dot{\otimes}} E)$, and differentiable vector functions.

Chapter 4, on special classes of spaces, considers inductive limits, direct sums, $\mathcal{LF}$ spaces, products anp direct sums of lines (including the connection with Ulam measures), metrizable spaces, weak* closed convex

subsets of the adjoint, continuity and openness of mappings for $\mathcal{F}$-spaces, $\mathcal{DF}$ spaces (excellent proofs), quasi-normable spaces, Schwartz spaces.

Chapter 5, compactness in topological vector spaces, contains the Krein-Milman theorem, compact operator theory for locally convex spaces, the Eberlein-Šmulian-Krein-Grothendieck theorems on weak compactness, the Dunford-Pettis result on L_1 compactness, and the R. S. Phillips theorem on weakly compact mappings of L_1 into a Banach space. Only the first of these topics is treated in the text of Bourbaki.

In conclusion (and, admittedly, with impertinence) I should like to observe that the author's style here, in contrast with that of some of his earlier work, is beginning to exhibit lucidity and restraint, so that the expository level begins to approach the scientific level of his communications. *J. L. Kelley* (Berkeley, Calif.).

Inaba, Mituo. On differential equations in locally convex spaces of some types. Kumamoto J. Sci. Ser. A. 2 (1955), 119–124.

L'auteur considère une équation différentielle $dx/dt = f(x, t)$, où t est réel, x prend ses valeurs dans un espace vectoriel topologique E, et f est définie dans $E\times I$ (I intervalle réel) et prend ses valeurs dans E. Il suppose que E est un espace de Fréchet et un espace de Montel, et que I est compact; dans ces conditions, si f est bornée et continue dans $E\times I$, le théorème de Peano se généralise, et l'équation différentielle admet une solution définie dans I, satisfaisant à des conditions intitiales données. La démonstration utilise la compacité des parties bornées de E et le théorème du point fixe de Tychonoff; elle est aussi valable sans hypothèse de métrisabilité sur l'espace de Montel E. *J. Dieudonné* (Evanston, Ill.).

Ōhira, Keishirō. Remarks on duality in linear spaces. Kumamoto J. Sci. Ser. A. 2 (1955), 125–128.

Let E be a locally convex Hausdorff linear space, and E^{*k} its dual equipped with the topology of uniform convergence on compact sets. If the natural map from E to the dual space of E^{*k} is onto, then E is called reflexive. It is proved that E is reflexive if and only if the closed, convex hull of every compact subset of E is compact. This result is applied to a paper by M. Smith [Ann. of Math. (2) **56** (1952), 248–253; MR **14**, 183]. [Reviewer's note: The author's main lemma follows immediately from the fact that, on an equicontinuous set of functions, the topologies of uniform convergence on compact sets and on finite sets coincide]. *E. Michael* (Seattle, Wash.).

Evgrafov, M. A. On a criterion for a basis. Dokl. Akad. Nauk SSSR (N.S.) **107** (1956), 199–201. (Russian)

Let

$$\mathfrak{A}(\sigma(r)) = \{f(x) \mid f \text{ measurable}, \int_0^\infty |f(x)| e^{-xr}dx = e^{O(\sigma(r))},\ r>0\}$$

where $\sigma(r)\to\infty$ more rapidly than $|\log 1/r|$ as $r\to 0$. A convergence topology is given to $\mathfrak{A}(\sigma(r))$ by: $f_N \Rightarrow 0$ as $N\to\infty$ in case $\int_0^\infty |f(x)|e^{-xr}dx \leq \varepsilon_N e^{\delta_N \sigma(r)}$, where ε_N, $\delta_N \to 0$ as $N\to\infty$. A set $\{\varphi_\lambda \mid 0\leq\lambda<\infty\}\subset\mathfrak{A}(\sigma(r))$ is called a basis in case there is for each $f\in\mathfrak{A}(\sigma(r))$ a unique function $a(\lambda)$ such that $f_N(x) = \int_0^N a(\lambda)\varphi_\lambda(x)dx \Rightarrow f(x)$ as $N\to\infty$. The author investigates criteria which qualify a set $\{\varphi_\lambda(x) \mid \varphi_\lambda(x)=0$ if $x<\lambda\}$ as a basis. The conditions are extremely involved and in part express the similarity between the functions $\varphi_\lambda(x)$ and the functions denoted by $\delta(x-\lambda)$. In context these latter seem to be the "Dirac" δ-functions or, more properly, approximate identities in $\mathfrak{A}(\sigma(r))$. *B. Gelbaum* (Minneapolis, Minn.).

Kasahara, Shouro. Sur un théorème de Gelfand. Proc. Japan Acad. 32 (1956), 131–134.

Let E be a locally convex Hausdorff space, and A an algebra of continuous, linear transformations on E, containing all one-dimensional ones, and carrying the topology of uniform convergence on a family of bounded subsets of E whose union generates E. It is proved, that if the set of invertible elements of A is open, then E and A are normable. Some results on the completeness of spaces of linear transformations are also obtained. *E. Michael.*

Charles, B. Espaces vectoriels topologiques. Ann. Univ. Sarav. 3 (1954), 360–371 (1955).

This is a readable exposition of several topics in the theory of linear topological spaces. The results are not new. *J. L. Kelley* (Berkeley, Calif.).

Pettis, B. J. Separation theorems for convex sets. Math. Mag. 29 (1956), 233–247.

This is an exposition of some of the basic properties of convex sets (in real topological linear spaces) which center around the separation theorem, and includes the separation theorem itself, the extreme point theorem, the "fundamental theorem" of game theory, and a bibliography of sixteen items. The principal novelty of treatment is that, wherever possible, only midpoint convexity is assumed. The few misprints should cause no trouble. *V. L. Klee, Jr.* (Los Angeles, Calif.).

Williamson, J. H. Two conditions equivalent to normability. J. London Math. Soc. 31 (1956), 111–113.

(1) Let E be locally convex, let L be a linear space of continuous transformations of E into a linear topological space F (with non-trivial topology), and suppose that L contains all continuous transformations of finite rank. If L can be given a topology such that the map $(x, f)\to f(x)$ of $E\times L$ into F is continuous, then E is normable.

(2) Let M be an algebra of continuous linear transformations of a locally convex space E into itself, and suppose that M contains all continuous transformations of finite rank. If composition in M is continuous relative to the topology of uniform convergence on some family of bounded subsets of E, then M and E are normable.

The author actually proves a slightly stronger form of (2). One version of (2) (for E) has recently been proved by A. Blair [Proc. Amer. Math. Soc. **6** (1955), 209–210; MR **16**, 935].

These results are obtained by extending an argument which may be phrased: If F is the adjoint of E and $(x, f)\to f(x)$ is bounded on a neighborhood $U\times V$ of 0 in $E\times F$, then V is equicontinuous, whence its polar in E is a bounded neighborhood of 0, and E is therefore normable. *J. L. Kelley* (Berkeley, Calif.).

See also: Ghika, p. 1107.

Banach Spaces, Banach Algebras

Suzuki, Noboru. On the invariants of W^*-algebras. Tôhoku Math. J. (2) 7 (1955), 177–185.

The author first develops in detail the way in which the possibility of representing a normal state as a finite

sum of vector states ties up with the coupling invariants of a finite w^* algebra. Then he defines the coupling invariants of a purely infinite w^* algebra, and proves that an algebraic isomorphism between two such w^* algebras is unitary if and only if the coupling invariants are carried across. *J. Feldman* (Princeton, N.J.).

Umegaki, Hisaharu. Positive definite function and direct product Hilbert space. Tôhoku Math. J. (2) **7** (1955), 206–211.

The author defines a positive definite function $V(\omega, \omega')$ from $\Omega\times\Omega$ to operators on a Hilbert space $\mathfrak{H}$ by the property $\sum_{i,j}(\xi_i, V(\omega_i, \omega_j)\xi_j)\geqq 0$ for $\xi_i\in\mathfrak{H}$, $\omega_i\in\Omega$. He also assumes that $V(\omega, \omega')=V(\omega', \omega)^*$, but this assumption is in fact redundant. A tensor product is defined between Ω and $\mathfrak{H}$, with inner product

$$\langle\sum_i\omega_i\otimes\xi_i, \sum_j\omega'_j\otimes\xi_j'\rangle=\sum_{i,j}(\xi_i, V(\omega_i, \omega_j')\xi_j');$$

taking quotients and completing, a Hilbert space $\Omega\otimes_V\mathfrak{H}$ is obtained. Specializing to the case where Ω is a group G, and $V(g, h)$ has the form $V_{h^{-1}g}$, a generalization of a theorem of Naimark is obtained; to wit, there is a unitary representation U_g of G into operators on $G\otimes_V\mathfrak{H}$ and a bounded linear transformation T: $\mathfrak{H}\to G\otimes_V\mathfrak{H}$ such that $T^*U_gT=V_g$. A "group algebra form" of the theorem is also given. Similar results had been obtained by W. Stinespring, but Umegaki remarks that his work was done independently; the connection between the two is seen by noting that if $\mathfrak{A}$ is a c^* algebra and $V(A, B)$ an operator-valued function on $\mathfrak{A}\times\mathfrak{A}$ which happens to have the form $W(B^*A)$ for some linear operator-valued function W, then V is positive-definite in the sense of Umegaki if and only if W is completely positive in the sense of Stinespring. *J. Feldman* (Princeton, N.J.).

Dieudonné, Jean. Sur la théorie spectrale. J. Math. Pures Appl. (9) **35** (1956), 175–187.

Suppose: E is a complex Banach space, E' is its dual, $\mathcal{L}(E)$ is the Banach algebra of continuous linear mappings of E into itself, $C(K)$ is the Banach algebra of continuous complex functions on a compact set K, $\mathcal{M}(K)$ is the Banach space of measures (continuous linear functionals) on K, and $\mathcal{B}(K)$ is the Banach algebra of bounded Borel functions on K. This paper is a study of representations of $C(K)$ in $\mathcal{L}(E)$. The notation and terminology regarding measures and integration is that of Bourbaki [Intégration, Chap. I–IV, Actualités Sci. Ind. no. 1175; Hermann, Paris, 1952; MR **14**, 960]. To start out with, let $f\to T_f$ be a continuous representation of $C(K)$ in $\mathcal{L}(E)$. Then, if $x\in E$, $x'\in E'$, $f\to\langle T_fx, x'\rangle$ is a continuous linear functional on $C(K)$, and hence is a measure $m_{x,x'}$. These measures are called the spectral measures associated with the representation. If we replace the continuous f by a bounded Borel function f, the mapping $x'\to\int f dm_{x,x'}$ is, for a fixed x, a continuous linear functional on E', hence an element of E'', which is also denoted by T_f. If E is reflexive it can be proved (a known result) that $f\to T_f$ is a continuous representation of $\mathcal{B}(K)$ in $\mathcal{L}(E)$. Here it is assumed directly, not that E is reflexive, but that the above consequence of reflexivity holds. In order that $T_fx=0$ it is necessary and sufficient that f be $|m_{x,x'}|$-negligible [i.e. that $f=0$ almost everywhere with respect to $|m_{x,x'}|$] for each x' in a set dense in E'.

Let E' (and hence E) be separable, with sequences (a_n') and (a_n) dense in the unit balls of E' and E, respectively. Let $\mu_{pq}=|m_{a_p,a_q'}|$. Let μ be a positive measure on K such that, for $N\subset K$, $\mu(N)=0$ is equivalent to "$\mu_{pq}(N)=0$ for each p and q." Then T_f depends only on the class $\tilde{f}$ of f in $L^\infty(\mu)$; we can write $\tilde{f}\to T_f$. This mapping is a topological isomorphism of the Banach algebra $L^\infty(\mu)$ onto a closed subalgebra in $\mathcal{L}(E)$. Contact is made here with recent work of Dunford [Pacific J. Math. **4** (1954), 321–354; MR **16**, 142]. In the remainder of the paper the algebra $L^\infty(\mu)$ is identified in a known way with the algebra $C(\tilde{K})$ for a certain Stone space $\tilde{K}$. Henceforth K is written for $\tilde{K}$, so that K is a Stone space in which a subset is non-dense, of first category, and of μ-measure zero if it is any one of these things. One can write $m_{a_p,a_q'}=g_{pq}\cdot\mu$, where g_{pq} is finite except on a μ-negligible set N_{pq}. Let $\Omega=K-N$, where N is the union of all the sets N_{pq}. By a 'field of vectors' on Ω is meant a mapping $\zeta\to\{g_q(\zeta)\}_{q=1,2,\dots}$, where each $g_q(\zeta)$ is a complex number. The vector $h=\{g_q\}$ is said to belong to $\mathcal{H}$ if each g_q is in $L^1(\mu)$ and if $\|h\|=\sup_q\int|g_q|d\mu<\infty$. Then $\mathcal{H}$ is a Banach space. The functions g_{pq} afford a natural means of constructing a topological isomorphism of E onto a subspace E_1 of $\mathcal{H}$ (a_p maps into $\mathbf{h}_p=\{g_{pq}\}_{q=1,2,\dots}$). If $f\in C(K)$ and $\mathbf{h}\in\mathcal{H}$, the mapping $\mathbf{h}\to f\mathbf{h}$ defines a continuous linear mapping U_f of $\mathcal{H}$ into itself, and $f\to U_f$ is an isometric representation of $C(K)$ in $\mathcal{L}(\mathcal{H})$. When E is identifeid with E_1, the restriction of U_f to E_1 is just T_f.

For a given ζ the dimension of the (algebraic) linear span of the set $\mathbf{h}_1, \mathbf{h}_2, \cdots$ is called the multiplicity $n(\zeta)$ of $f\to T_f$ at ζ. By the use of suitable projections, and partitioning of Ω, one can confine attention to the situation where $n(\zeta)$ is constant, $=n$, on Ω. It turns out that n is the smallest possible number of elements in a system $(\mathbf{a}_\alpha)$ of fields of vectors such that the linear span of $T_f\mathbf{a}_\alpha$ (α varying, f varying over $C(K)$) is dense in E. Let $\mathcal{A}$ be the image of $C(K)$ in $\mathcal{L}(E)$ under the mapping $f\to T_f$ and let $\mathcal{A}'$ be the set of elements which commute with all elements of $\mathcal{A}$. Then $n=1$ implies $\mathcal{A}'=\mathcal{A}$. Here contact is made with work of Bade [Pacific J. Math. **4** (1954), 393–413; Bull. Amer. Math. Soc. **61** (1955), 137; MR **16**, 144]. When E is reflexive and $n=1$, E is isomorphic to the Köthe space of μ-measurable f such that $\sup_q\int|fa_{1q}|d\mu<\infty$ ($\mathbf{a}_1=\{a_{1q}\}$ being a generator of E). The paper concludes with a discussion of counterexamples. *A. E. Taylor.*

Dieudonné, Jean. Champs de vecteurs non localement triviaux. Arch. Math. **7** (1956), 6–10.

This paper is based on the author's study of spectral multiplicity theory in Banach spaces [cf. the preceding review]. Its essential purpose is to show by counterexample that in the case of a constant finite multiplicity the situation can be much more complicated than when the basic space is a Hilbert space. That is, if the multiplicity is k, the Banach space need not be the direct sum of k invariant subspaces on each of which the multiplicity is 1. This need not be true even in a 'local' sense explained in the paper.

A. E. Taylor (Los Angeles, Calif.).

Miyanaga, Yasue. A note on Banach algebras. Proc. Japan Acad. **32** (1956), 176.

A proof of a theorem due to the reviewer [Proc. Cambridge Philos. Soc. **47** (1951), 473–474; MR **13**, 256], concerning real or complex Banach algebras, for the complex case only. The proof is the one expected on the basis of this restriction, depending on holomorphy and the Liouville theorem for vector-valued functions.

R. E. Edwards (London).

Akilov, G. P. On extension of linear operations. Leningrad. Gos. Univ. Uč. Zap. **144**. Ser. Mat. Nauk 23 (1952), 47–84. (Russian)

This paper gives proofs for the notes in Dokl. Akad. Nauk SSSR (N.S.) **57** (1947), 643–646; **59** (1948), 417–418 [MR **9**, 241, 358]. The problem is to characterize those Banach spaces Y with Property (E): for every linear subspace of a normed linear space, $X_0 \subseteq X$, and for every linear operator U_0 from X_0 to Y there is an extension U of U_0 defined on all X so that $\|U\|=\|U_0\|$. This is shown to be equivalent to several other properties of Y [see Goodner, Trans. Amer. Math. Soc. **69** (1950), 89–108; MR **12**, 266]. It is proved that the space of continuous functions on a compact Hausdorff space M has property (E) if the space is extremally disconnected. Examples of spaces without (E) are given. This paper was written before Kelley [ibid. **72** (1952), 323–326; MR **13**, 659] had shown that the only spaces with property (E) were the examples given above. *M. M. Day* (Urbana, Ill.).

Rutickiĭ, Ya. B. On a property of completely continuous linear integral operators operating in Orlicz spaces. Uspehi Mat. Nauk (N.S.) **11** (1956), no. 2(68), 201–208. (Russian)

In the usual Orlicz space context and notations, let E_M be the closure in L_M^* of the bounded functions. A sequence $\{u_n\} \subset L_M^*$ is called (o)-weakly convergent in case $\int u_n v\,dx$ converges for all $v \in E_N$. Parallel to the classical topological linear space theorem the author shows: A (o)-weakly continuous functional l on L_M^* is characterized by an equation: $l(u)=\int uv\,dx$ $(v \in E_N)$. If $K(x, y)$ is such that $|\int K(x, y)u(x)v(y)\,dxdy|<\infty$ for all $u \in L_{M_1}^*$ and $v \in L_{N_2}^*$, then the operators $A(u)=\int K(x, y)u(y)\,dy$ and $A^*(v)=\int K(x, y)v(y)\,dy$ map $L_{M_1}^*$ (resp. $L_{N_2}^*$) into $L_{M_2}^*$ (resp. $L_{N_1}^*$), (M_i, N_i) are conjugate pairs. If A is completely continuous then A carries (o)-weakly convergent sequences into strongly convergent ones, if and only if A^* carries E_{N_2} into E_{N_1}. Finally, if f maps an open sphere T of $L_{M_2}^*$ into L_{M_1}, then, for $u_n \in T$, $u_n \to u_0$, $f(u_n) \to f(u_0)$ in the (o)-weak sense.
B. Gelbaum (Minneapolis, Minu.).

Luxemburg, W. A. J.; and Zaanen, A. C. Conjugate spaces of Orlicz spaces. Nederl. Akad. Wetensch. Proc. Ser. A. **59**=Indag. Math. **18**(1956), 217–228.

If Φ and Ψ are complementary Young functions, and if $M_\Phi(f)=\int \Phi(|f|)d\mu$, $\|f\|_{M\Phi}=\inf\{k^{-1}|M_\Phi(kf)\leq 1\}$, then the set $L_{M\Phi}$ of f such that $\|f\|_{M\Phi}<\infty$ is a Banach space. If $L_\Phi=\{f|\sup \int|fg|d\mu=\|f\|_\Phi<\infty|\ \|g\|_{M\Psi}\leq 1\}\}$, then the set L_Φ and $L_{M\Phi}$ are identical but in general $\|f\|_{M\Phi}\leq\|f\|_\Phi\leq 2\|f\|_{M\Phi}$. L_Φ is called the associate space of $L_{M\Psi}$ and the associate space of L_Φ is again $L_{M\Psi}$. Furthermore, $L_\Phi \subseteq L_{M\Psi}^*$, $L_\Psi \in L_{M\Phi}^*$ with equality if and only if $L_{M\Phi}$ and $L_{M\Psi}$ are reflexive.

The present paper solves the problem of extending the functionals M_Φ, $\|\cdots\|_{M\Phi}$ and $\|\cdots\|_\Phi$ from L_Φ to $L_{M\Phi}^*$ with preservation of the obvious functional relations. A crucial result is: If $g \in L_\Psi$, then

$$M_\Psi(g)=\sup\{|\int gd\mu|-M_\Phi(f)|f \in L_\Phi\};$$

the dual is also valid. From this, the natural extension of M_Ψ to $L_{M\Phi}^*$ is via the equation

$$M_\Psi(f^*)=\sup\{|f^*(f)|-M_\Phi(f)|\ f \in L_\Phi\}.$$

Similar devices are used to extend $\|\cdots\|_{M\Phi}$ and $\|\cdots\|_\Phi$.

There is also a discussion of circumstances under which convergence in the obvious "weak" topologies takes place.
B. Gelbaum (Minneapolis, Minn.).

Hirschman, Isidore. Sur les polynomes ultrasphériques. C. R. Acad. Sci. Paris **242** (1956), 2212–2214.

The author sets up a semi-simple commutative Banach algebra of functions f on the interval $(-1, +1)$ in such a way that if φ is a multiplicative linear functional on this algebra, $\varphi(f)$ can be expressed as the integral of the product of f and of a certain ultraspherical polynomial. In this way, a generalisation of Wiener's theorem (that an analytic function of a function with absolutely convergent Fourier series has itself an absolutely convergent Fourier series) to certain absolutely convergent series of ultraspherical polynomials is obtained. Corresponding results are given for a corresponding algebra of functions of an integer. *J. Schwartz* (New York, N.Y.).

Arens, Richard; and Hoffmann, Kenneth. Algebraic extension of normed algebras. Proc. Amer. Math. Soc. **7** (1956), 203–210.

Let A be a commutative normed linear algebra over the complex field, with a unit. Let $\alpha(x)=a_0+a_1x+\cdots+a_{n-1}x^{n-1}+x^n$, where $a_i \in A$ $(0\leq i\leq n-1)$, and $n\geq 2$. The basic problem is to obtain an isometric extension B of A which contains a root of $\alpha(x)=0$; this is achieved by a straightforward construction. The algebra of equivalence-classes $B=A[x]/(\alpha(x))$ can be given a norm with the required properties by writing

$$\|\bar{\beta}\|=\|b_0\|+\|b_1\|t+\cdots+\|b_{n-1}\|t^{n-1},$$

where $\beta(x)=b_0+b_2x+\cdots+b_{n-1}x^{n-1}$ is the unique polynomial of degree $<n$ in the equivalence-class $\bar{\beta}$, and t is a positive real number such that $t^n\geq\|a_0\|+\|a_1\|t+\cdots+\|a_{n-1}\|t^{n-1}$. If A is a Banach algebra, so is B.

The algebra A is called tractable if the intersection of its closed maximal ideals is zero. If A is tractable, and the discriminant of $\alpha(x)$ is not a zero divisor, then B is tractable. In any case a tractable extension of A is obtained by taking the quotient of B by the intersection of its closed maximal ideals.

The above results obviously hold also if $\alpha(x)$ is replaced by a polynomial in which the coefficient of the highest power of x has an inverse. Misprint: page 207, line 12, for $\lambda(x)$ read $\gamma(x)$. *J. H. Williamson* (Belfast).

de Leeuw, K.; and Mirkil, H. Intrinsic algebras on the torus. Trans. Amer. Math. Soc. **81** (1956), 320–330.

The material of this paper is closely connected with, and generalises, work of G. E. Šilov [in particular, Uspehi Mat. Nauk (N.S.) **6** (1951), no. 1(41), 91–137; MR **13**, 139; **17**, 512]. By an intrinsic homogeneous algebra A on the n-dimensional torus T_n the authors mean a commutative semi-simple complex Banach algebra of continuous functions on T_n such that (1) the maximal ideal space of A is T_n; (2) A is invariant under all translations and automorphisms of T_n; (3) for each $f \in A$ the map $x \to f_x$ from T_n to A is continuous (where $f_x(y)=f(y-x)$). The algebra is called quasi-analytic if the vanishing of a function on a non-void open set implies that it is identically zero; it is regular if any two disjoint closed sets can be separated by an element of the algebra. Theorem 1: A is either quasi-analytic or regular. Theorem 3: If A is of type C, in the sense of Šilov, and contains the algebra of infinitely-differentiable functions, then for some integer m, A is the algebra of m-differentiable functions. This was proved by Šilov [loc. cit.] for the case $n=1$. The proof of Theorem 3 requires the authors to investigate some purely algebraic properties of formal power series; in this connection they prove Theorem 2: If K is a field of

characteristic zero, and I is a proper ideal in $K[[X_1, \cdots, X_n]]$, invariant under unimodular substitutions, then I is some power of the unique maximal ideal. *J. H. Williamson* (Belfast).

Shimoda, Isae. Notes on general analysis. V. Singular subspaces. J. Gakugei Tokushima Univ. Nat. Sci. Math. **6** (1955), 5–18.

[For parts I–IV see same J. **2** (1952), 13–20; **3** (1953), 12–15; **4** (1954), 1–10; **5** (1954), 1–7; MR **14**, 766; **15**, 38, 801; **16**, 1123.] Let L_0 be a closed linear subspace of defect at least 2 in a complex Banach space E_1. Let f on E_1-L_0 to a second complex Banach space E_2 be analytic and homogeneous of positive integral degree n on E_1-L_0. Then f is a homogeneous polynomial of degree n. A function of this kind cannot exist if n is a negative integer. The first assertion is false if the defect of L_0 is 1. A study is made of homogeneous analytic functions on E_1-L_0 when the defect of L_0 is 1. Next it is shown that a closed linear manifold of defect 2 or more cannot contain all the singularities of a function which is otherwise analytic on E_1. There are some theorems about functions analytic outside a closed linear manifold of defect 1.

A. E. Taylor (Los Angeles, Calif.).

See also: Boas, p. 1080; Pinsker, p. 1052.

Hilbert Space

Kato, T.; and Taussky, O. Commutators of A and A^*. J. Washington Acad. Sci. **46** (1956), 38–40.

Let A and B be bounded linear operations on a Hilbert space X and define the commutator $[A, B]=AB-BA$. Putnam has shown for finite dimensional X that $[A, [A, A^*]]=0$ implies $[A, A^*]=0$, i.e. A is normal. This paper records four alternate proofs, extends one of them to a general Hilbert space X, and even proves for general X that $[[A, A^*], [A, [A, A^*]]]=0$ implies $[A, A^*]=0$.

F. H. Brownell (Seattle, Wash.).

Beck, W. A.; and Putnam, C. R. A note on normal operators and their adjoints. J. London Math. Soc. **31** (1956), 213–216.

The following theorem is proved: I. Let A, N be bounded operators in Hilbert space, N normal, $AN=N^*A$. If the canonical resolution of A is PU (P positive definite, U unitary), where A^{-1} is bounded, and if the spectrum of U is contained in some open semi-circle of $|Z|=1$, then $N=N^*$. II. (same notations): If $AN=N^*A$ and if for Im $(Z)\neq 0$, either Z or $\bar{Z} \notin$ spectrum of N, then $AN=NA$. The authors note that $AN=N^*A$ and $AN=NA$ imply $N=N^*$ if A^{-1} is bounded. The proof flows easily from the spectral decomposition of U.

B. Gelbaum (Minneapolis, Minn.).

Harazov, D. F. On spectral decompositions of certain linear operators. Dokl. Akad. Nauk SSSR (N.S.) **102** (1955), 693–696. (Russian)

In an earlier paper [same Dokl. (N.S.) **91** (1953), 1023–1026; MR **15**, 881], the author proved some expansion theorems with respect to solutions of the equation

$$(E-\lambda A_1-\lambda^2 A_2)x=0,$$

where A_1 and A_2 are symmetrisable operators of finite norm in Hilbert space, and E is the identical operator. In the present paper it is shown that these results remain valid if weak convergence is replaced by strong convergence. A similar improvement is made in the results of another paper [Akad. Nauk Gruzin. SSR. Trudy Tbiliss. Mat. Inst. Razmadze **20** (1954), 297–315; MR **16**, 934].

F. Smithies (Cambridge, England).

Milkman, Joseph. Hermite polynomials, Hermite functionals and their integrals, in real Hilbert space. Riv. Mat. Univ. Parma **6** (1955), 65–88.

This paper gives the details of Friedrichs' construction of a finitely additive, nonnegative set function over the Borel sets of real separable Hilbert space. The construction uses Hermite polynomials in a natural way as suggested by quantum field theory [Friedrichs, Comm. Pure Appl. Math. **4** (1951), 161–224; MR **13**, 520]. The resulting set function is not σ-additive, and thus not a measure. *F. H. Brownell* (Seattle, Wash.).

Segal, I. E. Tensor algebras over Hilbert spaces. II. Ann. of Math. (2) **63** (1956), 160–175.

This paper is a sequel to part I [Trans. Amer. Math. Soc. **81** (1956), 106–134; MR **17**, 880], which gave the equivalence of the algebra of symmetric tensors over a complex Hilbert space X with the L_2 space over a real Hilbert space X', of which X is the complexification, according to an integration theory on X' there developed in terms of a normal distribution. In this paper the algebra of skew-symmetric tensors is similarly found to be equivalent to the L_2 space over X' with the normal distribution replaced by what the author calls the Clifford distribution.

F. H. Brownell (Seattle, Wash.).

Nevanlinna, Rolf. Über metrische lineare Räume. V. Relationen zwischen verschiedenen Metriken. Ann. Acad. Sci. Fenn. Ser. A. I. no. **222** (1956), 6 pp.

In further elaboration of his program [same Ann. no. **163** (1954); MR **15**, 717] the author outlines a method for finding, in a class (H) of equivalent positive definite Hermitian majorant metrics for a generalized Hilbert space R, the minimal majorant P, for a given semi-definite form Q: If H_1 and H_2 belong to (H), they engender two unique operators H_{12} and H_{21} such that (i) $H_1(x, y)=H_2(H_{21}x, y)=H_2(x, H_{21}y)$ and $H_2(x, y)=H_1(H_{12}x, y)=H_1(x, H_{12}y)$; (ii) $H_{12}H_{21}=I$, (iii) if H_1, H_2, $H_3 \in (H)$, then $H_{13}=H_{12}H_{23}$. In (H) there are subclasses corresponding to different decompositions of R. The operators of these subclasses are characterized by being commutative with each other. Two distinct abelian groups of such operators engender different minimal majorants P, and there is no smallest such since they "cross over" ("durchkreuzen") in pairs.

There is a short discussion of the circumstances under which two forms Q_1, Q_2 permit a spectral association: $Q_1(x, x)=\int_{-\infty}^{\infty} \lambda dQ_2(E_\lambda x, E_\lambda x)$. *B. Gelbaum.*

See also: Fišman, p. 1070; Nigam and Foldy, p. 1163; Sommer and Mehring, p. 1071; Umegaki, p. 1112.

TOPOLOGY

Have

★**Patterson, E. M. Topology.** Oliver and Boyd, Edinburgh and London; Interscience Publishers, Inc., New York, 1956. viii+128 pp. $1.55.

The author intended this booklet for the use of students taking an "Honours Course" in topology at a British University. As such, it is written on a more elementary level than any of the existing textbooks on topology [with the possible exception of P. Alexandroff, Einfachste Grundbegriffe der Topologie, Springer, Berlin, 1932]. On the one hand, the author has avoided "long and difficult proofs which might distract the reader's attention from the development of the theory," or a discussion of those "sections of modern topology which seem too deep for the beginner to appreciate." On the other hand, he has "included details which might be omitted in a more advanced treatise, preferring to sacrifice brevity to intelligibility." It is the author's purpose "that the interested reader will gain from it sufficient knowledge to enable him to proceed to more advanced treatises with confidence."

There are six chapters. Chapter I, the Introduction, contains an informal, intuitive discussion of the following topics: Topological equivalence, surfaces, orientability of surfaces, connectivity of spaces, topological invariants of spaces, Euler's Theorem for polyhedra, and the coloring of maps on surfaces. No attempt is made to be rigorous in this chapter. Chapter II, entitled "Topological Spaces", starts with a summary of some of the basic definitions and notations of set theory. Then the topology of Euclidean n-space is introduced, leading to the general notion of a metric space. Finally, the usual definition of a topology on a space in terms of open sets is given, and the basic properties of open and closed sets, etc. are proved. There are also brief sections on identification spaces, product spaces, and topological groups.

Chapter III, on "Particular Types of Topological Spaces," discusses Hausdorff spaces, normal spaces, and the notions of completeness, compactness, and connectivity. A complete proof of Urysohn's famous lemma on the separation of closed subsets of a normal space by continuous real-valued functions is given; this is probably the most difficult proof in the entire book.

Chapters IV, V, and VI are intended to give an introduction to algebraic topology. Chapter IV, entitled "Homotopy," gives the definitions of homotopy of continuous mappings and homotopy type of spaces. The fundamental group is introduced and its basic properties are proved. Finally, there is a brief section devoted to the definition of the higher homotopy groups of a space. Chapter V is devoted to the geometry of simplicial polyhedra in Euclidean space. This chapter is preparatory to chapter VI, entitled "Homology." Chapter VI begins with a discussion of the fundamental properties of finitely generated abelian groups. Some of the properties are proved in full detail, while the author merely gives a reference for the proof of the fundamental theorem. Then the integral chain groups and the boundary operator of a simplicial polyhedron are defined, leading to the usual definitions of homology groups, Betti numbers, and torsion coefficients. There is a briefer discussion of homology with more general coefficients and cohomology groups. No attempt is made to prove that the homology groups of a simplicial polyhedron are topological invariants; the reader is referred to more advanced treatises for the proof.

At each stage of the development the author is careful to provide adequate motivation for the introduction of new ideas and definitions. There are plenty of examples throughout the text, many of which are explained with the help of diagrams. At the end of each chapter there is a list of exercises.

This booklet goes a long way toward filling a gap in the literature which must have been rather acutely felt by anybody who has ever tried to teach a beginning course in topology, especially of the algebraic variety. It could be used as a text at those universities that give a topology course to advanced undergraduates. Even more important, the average graduate student could profitably work through it himself before taking a first course in topology.

The reviewer noticed a couple of errors. In exercises 3 on page 88, the reader is asked to "show that the fundamental group of the Klein Bottle is the direct product of an infinite cyclic group and a cyclic group of order two." On page 104, the definition of an abelian group's being finitely generated is not correctly worded. To get the correct statement, a phrase containing an existential quantifier must be added.

W. S. Massey.

Fort, M. K., Jr. Category theorems. Fund. Math. **42** (1955), 276–288.

The author gives a unified treatment of a remarkable collection of category theorems, basing his work on the following general result: If T and T^* are topologies for Y, then each T^*-continuous mapping of an arbitrary topological space X into Y is T-continuous at the points of a residual subset of X provided the following condition (α) is satisfied: there are sequences $\{U_n\}$ and $\{K_n\}$ of subsets of Y such that $U_n \subset K_n$ for each n, for every T-neighborhood U of each point p there is n such that $p \in U_n \subset K_n \subset U$, and if $q \in U_n$ then there is a T^*-open set V such that $q \in V$ and $V \sim K_n$ is T^*-open. More or less immediate consequences of this result include the real variable theorem on points of continuity of a semi-continuous function, the Osgood-Baire theorem on pointwise limits of sequences of continuous functions, a theorem deducing continuity on a residual subset of a product space from continuity in each variable separately, and a related result concerning continuity of operation of a group of homeomorphisms of a topological space.

The author defines a topology T^* (called the overlap topology) for the space of all continuous real valued functions of a real variable which has the properties: differentiation is a continuous mapping of the continuously differentiable functions into the continuous functions, and T^* and the compact-open topology T satisfy (α). From these considerations he deduces the following stronger form of a theorem of Montgomery's: If G is a second category group which is a transformation group of a manifold of class C^1 and if each member of G is C^1, then the derivatives of the functions which define the transformations locally are continuous in all variables simultaneously. A theorem concerning the space of infinitely differentiable functions is also obtained.

Other results include: If T is the norm topology for $L^p[0, 1]$ and T^* is the topology of convergence in measure, then T and T^* satisfy (α), and certain "upper" and "lower" topologies for the space of all measurable sets also satisfy (α). If f is a map of a topological space X into the space

of all subsets of a separable metrizable space S, and if f is lower semi-continuous (or upper semi-continuous and S locally compact), then there is a residual set at each point of which f is upper semi-continuous (lower semi-continuous respectively). (Semi-continuity of set valued functions is defined in terms of convergence.) *J. L. Kelley.*

Deprit, André. Sur les $\mathcal{M}$-compactifications d'Alexandroff. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 266–269.

Some elementary results on one-point $\mathcal{M}$-compactifications, generalizing analogous results on one-point compactifications by G. Papy [same Bull. (5) **39** (1953), 937–941, MR **15**, 546]. (For a cardinal number $\mathcal{M}$, a space is $\mathcal{M}$-compact if every open covering with $\leq\mathcal{M}$ elements has a finite subcovering). *E. A. Michael.*

Tanaka, Tadashi. On the family of connected subsets and the topology of spaces. J. Math. Soc. Japan **7** (1955), 389–393.

The author investigates conditions under which the topology of a set is determined by its connected subsets. Let A be a set of points and C be a collection of subsets of A. Then there do not exist different topologies for the set A under which C is the collection of all connected subsets of A and such that A is separable metric and either (1) semi-locally connected or (2) compact and satisfies the condition (*). The condition (*) is that for each point p of A and sequences $\{p_n\}$ of points of A converging to p, there exist a point o, a subsequence $\{p_{n_i}\}$ of $\{p_n\}$ and a sequence $\{op_{n_i}\}$ of arcs converging to an arc op.

E. Dyer (Baltimore, Md.).

★ **Kramer, Eugen. Studien zur topologischen Axiomatik der Bewegungsgruppen.** Dissertation, Eidgenössische Technische Hochschule in Zürich, 1955. 46 pp.

Let Γ be a group of homeomorphisms of a space R. A metric of R invariant under Γ will be called a Γ-metric. This paper is concerned with the existence theorems of Γ-metrics. Necessary and sufficient conditions for the Γ-metrisability (as well as convex Γ-metrisability) of R are obtained. Using these results, the author gives a topological characterization of two-point homogeneous metric spaces. The main theorem can be stated as follows: "Suppose that (a) R is a connected, locally compact, Hausdorff space satisfying the first countability axiom and Γ a group of homeomorphisms of R; (b) R has a point p with the following property: to each neighborhood $W(p)$ of p, there exists a neighborhood $V(p)$ such that $V(p)\cup\sigma V(p)\subset W(p)$ whenever $V(p)\cap\sigma V(p)\neq 0$, $\sigma\in\Gamma$; and (c) for any two group-spheres with the same center, one of them must separate the other from the center (by a group-sphere with center q, we mean the orbit of a point under the subgroup $\{\sigma:\sigma\in\Gamma, \sigma(q)=q\}$). Then R has a convex Γ-metric such that Γ is transitive on equi-distant pairs of points of R.

This theorem has some contacts with a paper of J. Tits [Bull. Soc. Math. Belg. **1952**, 44–52; MR **15**, 334].

H. C. Wang (Seattle, Wash.).

Iséki, Kiyoshi. Notes on topological spaces. II. Some properties of topological spaces with Lebesgue property. Proc. Japan Acad. **32** (1956), 171–173.

[For definitions, see MR **17**, 389.] Let f and g be, respectively, upper and lower semi-continuous real-valued functions on a uniform space X, every countable covering of which has the Lebesgue property, and let $f(x)<g(x)$ for all x in X. Then for every $\varepsilon>0$ there exists a vicinity V for X such that $f(x')<g(x'')+\varepsilon$ whenever $(x', x'')\in V$. [The paper's weaker assumption that $f(x)\leq g(x)$ is a misprint.] *E. A. Michael* (Seattle, Wash.).

Iséki, Kiyoshi. On the property of Lebesgue in uniform spaces. VI. Proc. Japan Acad. **32** (1956), 117–119.

[For definition of the Lebesgue property, see the review of parts I–V, same Proc. **31** (1955), 220–221, 270–271, 441–442, 524–525, 618–619; MR **17**, 389.] In this paper it is proved that if S is a uniform space where every finite covering has the Lebesgue property, and if its completion $\hat{S}$ is normal, then $\dim S=\dim\hat{S}$ (where dim denotes Lebesgue dimension). *E. Michael.*

Denjoy, Arnaud. Les ensembles parfaits cartésiens totalement discontinus. C. R. Acad. Sci. Paris **242** (1956), 2195–2198.

A proof is given for the well known theorem that every totally disconnected perfect set P in a Euclidean space E lies on a simple arc in E. Historical remarks concerning this theorem may be found in a paper by Moore and Kline [Ann. of Math. (2) **20** (1918), 218–223], who proved (in a plane) that the same conclusion holds if P, instead of being totally disconnected, has as non-degenerate components only simple arcs C no interior point of which is a limit point of $P-C$. It was shown by the reviewer [Fund. Math. **18** (1932), 47–60] that the theorem in original form as above holds in case E, instead of being a Euclidean space, is any locally connected continuum having no local separating point. *G. T. Whyburn.*

Scorza Dragoni, Giuseppe. Un'osservazione sul lemma di Sperner. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 204–206.

The author reproduces one of the many proofs that a solid sphere can not be retracted into its boundary.

P. A. Smith (New York, N.Y.).

Trucco, Ernesto. A note on the information content of graphs. Bull. Math. Biophys. **18** (1956), 129–135.

Ore, Oystein. Studies on directed graphs. I. Ann. of Math. (2) **63** (1956), 383–406.

Let the number of edges of a directed graph G leaving or entering a vertex v be $\alpha(v)$ or $\alpha^*(v)$ respectively, and let the number entering v from a set A of vertices be $\varrho_A{}^*(v)$. The author supposes integers $\varkappa(v)$ and $\varkappa^*(v)$, satisfying $0\leq\varkappa(v)\leq\alpha(v)$ and $0\leq\varkappa^*(v)\leq\alpha^*(v)$ to be defined at each vertex v. He writes $\mu_A{}^*(v)=\min(\varkappa^*(v), \varrho_A{}^*(v))$ and $\delta(A)=\sum\varkappa(v)-\sum\mu_A{}^*(\omega)$. In the latter formula v runs through the vertices of A and ω through the vertices entered by edges from A. Note $\delta(A)\geq 0$.

The author calls δ a deficiency function. There is a corresponding deficiency function δ^* obtained by using $\varkappa^*(v)$ instead of $\varkappa(v)$ and taking the edges in reverse directions. There are two more, $\bar{\delta}$ and $\bar{\delta}^*$ obtained by replacing $\varkappa(v)$ by $\alpha(v)-\varkappa(v)$ and $\varkappa^*(v)$ by $\alpha^*(v)-\varkappa^*(v)$. The bulk of the paper consists of a detailed theory of the deficiency functions and the relations between them.

The author has in mind the problem (for finite graphs) of finding conditions for the existence of a subgraph having just $\varkappa(v)$ edges leaving and $\varkappa^*(v)$ entering each vertex v. It is easily seen to be necessary that $\delta(A)=\delta^*(A)=0$ for each set of vertices. The author shows that this condition, when combined with the obvious requirement $\sum\varkappa(v)=\sum\varkappa^*(v)$, is sufficient.

G is called regular of degree n if $\alpha(v)=\alpha^*(v)=n$ for each v. A factor of G of degree n is a regular subgraph of G of degree n which includes all the vertices of G. Among other results the author deduces from his main theorem that every finite regular directed graph has a factor of degree 1.

W. T. Tutte (Toronto, Ont.).

See also: Curtis, p. 1046; Goetz, p. 1107; Schwarz, p. 1107; Shirai, p. 1065.

Algebraic Topology

★**Séminaire Henri Cartan de l'Ecole Normale Supérieure, 1950/1951. Cohomologie des groupes, suite spectrale, faisceaux. 2e éd.** Secrétariat mathématique, 11 rue Pierre Curie, Paris, 1955. 188 pp.

The publication of the material covered by the Cartan seminar in 1950/51 has been long overdue. The first seven exposés are devoted to the homology and cohomology theory of groups and applications of the theory, principally to simple algebras. This material must surely be contained in the book "Homological algebra" [Princeton, 1956] by Cartan and Eilenberg, where it would be presented within the more general framework of the homology and cohomology theory of modules. The present treatment lacks the explicit notions of a projective module and of an injective resolution of a module; nor is a construction given for embedding a module in an injective module.

The material of exposés 8-10 on spectral sequences and their application to fibre spaces will be familiar to all who have read Serre's thesis [Ann. of Math. (2) **54** (1951), 425–505; MR **13**, 574]. Exposés 11–13 are concerned with the Cartan-Leray method for attaching a spectral sequence to a covering (or, more generally, a space with operators). This technique is referred to by Serre [loc. cit.] and has been used — and, essentially, described — by Hochschild and Serre [Trans. Amer. Math. Soc. **74** (1953), 110–134; MR **14**, 619].

Exposés 14-21 have already been reviewed [MR **14**, 670]. The reviewer welcomes this opportunity to correct an error which appeared on line 39 of that review; namely, for 'in supposing that all closed neighbourhoods of members of Φ belong to Φ', read 'in supposing that all members of Φ have a closed neighborhood belonging to Φ'.

P. J. Hilton (Manchester).

James, I. M. On the suspension triad. Ann. of Math. (2) 63 (1956), 191–247.

L'auteur étudie la „suite exacte de suspension" d'un espace A avec point-base:

$$(1)\quad \cdots\to\pi_r(A)\xrightarrow{E}\pi_{r+1}(\hat{A})\xrightarrow{i}\pi_{r+1}(\hat{A};C_+,C_-)\xrightarrow{\Delta}\pi_{r-1}(A)\to\cdots,$$

où $\hat{A}$ désigne la suspension de A, E l'homomorphisme de suspension, C_+ et C_- les deux cônes dont $\hat{A}$ est la réunion et A l'intersection [cf. Blakers-Massey, Ann. of Math. (2) **53** (1951), 161–205; MR **12**, 435]; il introduit un automorphisme involutif τ de la suite (1): τ est l'identité sur $\pi_r(\hat{A})$, et sur $\pi_{r+1}(\hat{A})$ est opposé à l'automorphisme induit par la symétrie de $\hat{A}$ par rapport à A. Il étudie le "triad Whitehead product"

$$\pi_p(A)\times\pi_q(A)\to\pi_{p+q+1}(\hat{A};C_+,C_-)$$

égal au signe près à celui défini par Blakers et Massey [ibid. **58** (1953), 295–324; MR **15**, 731]; si $\{\beta,\gamma\}$ désigne le „triad Whitehead product" de $\beta\in\pi_p(A)$ et $\gamma\in\pi_q(A)$, on a $\Delta\{\beta,\gamma\}=[\beta,\gamma]$, produit de Whitehead ordinaire. L'auteur donne enfin une nouvelle définition (§ 8) de la construction de Hopf, qui à chaque application f d'un produit de sphères $S_p\times S_q\to A$ associe $c(f)\in\pi_{p+q+1}(\hat{A})$; il prouve que les éléments de $\pi_{p+q+1}(\hat{A})$ qui peuvent être obtenus par une construction de Hopf de type (β,γ) sont exactement ceux que i envoie sur l'élément $\{\beta,\gamma\}$. On a une identité de Jacobi

$$(-1)^{rp}\{[\beta,\gamma],\delta\}+(-1)^{pq}\{[\gamma,\delta],\beta\}+(-1)^{qr}\{[\delta,\beta],\gamma\}=0$$

pour $\beta\in\pi_p(A)$, $\gamma\in\pi_q(A)$, $\delta\in\pi_r(A)$; une loi de commutation

$$\{\beta,\gamma\}=(-1)^{pq}\{\gamma,\beta\}+(-1)^{pq}[E\beta,E\gamma];$$

on explicite l'effet de l'automorphisme τ sur le "triad Whitehead product": $\tau\{\beta,\gamma\}=(-1)^{pq}\{\gamma,\beta\}$, et l'effet de τ sur la construction de Hopf.

La suite exacte (1), munie de toutes les opérations précédentes, est un foncteur covariant de l'espace A. On définit un isomorphisme naturel de cette suite exacte avec deux autres suites exactes:

$$(2)\quad \cdots\to\pi_r(A)\to\pi_r(\Omega(\hat{A}))\to\pi_r(\Omega(\hat{A}),A)\to\pi_{r-1}(A)\to\cdots,$$
$$(3)\quad \cdots\to\pi_r(A)\to\pi_r(A_\infty)\to\pi_r(A_\infty,A)\to\pi_{r-1}(A)\to\cdots,$$

où $\Omega(\hat{A})$ désigne l'espace des lacets de $\hat{A}$, et A_∞ désigne le "reduced product space" de A, étudié dans un article antérieur de I. M. James [ibid. **62** (1955), 170–197; MR **17**, 396]. Dans cet article on a établi l'isomorphisme des suites (2) et (3) sous l'hypothèse que A est un CW-complexe dénombrable avec un seul sommet (le rapporteur n'a pas l'impression que cette restriction soit essentielle; on doit naturellement supposer A connexe). Les isomorphismes de (1) avec (2) et (3) permettent de donner de nouvelles interprétations des opérations étudiées, notamment du "triad Whitehead product" (th. 7.1). De plus la filtration naturelle de A_∞ permet de définir une filtration intéressante sur les groupes d'homotopie $\pi_{r+1}(\hat{A})$ et $\pi_{r+1}(\hat{A};C_+,C_-)$. Enfin l'isomorphisme de (1) et (3) permet de généraliser l'„invariant de Hopf": on définit un homomorphisme h: $\pi_r(A_\infty,A)\to\pi_r(A_\infty')$, où A' désigne le quotient de $A\times A$ obtenu en concentrant en un seul point sous-espace $A\vee A$; h devient un homomorphisme $\pi_{r+1}(\hat{A};C_+,C_-)\to\pi_{r+1}(A*A)$, où $A*A$ désigne le joint de A avec lui-meme. Si $\beta\in\pi_p(A)$, $\gamma\in\pi_q(A)$, alors $h\{\beta,\gamma\}=\beta*\gamma$. On définit

$$H=h\circ i:\pi_{r+1}(\hat{A})\to\pi_{r+1}(A*A);$$

si $\alpha\in\pi_{p+q+1}(\hat{A})$ est obtenu par une construction de Hopf de type (β,γ), on a $H(\alpha)=\beta*\gamma$. Dans un travail ultérieur, l'auteur étudiera l'homomorphisme H lorsque A est la sphère S_n; alors H: $\pi_{r+1}(S_{n+1})\to\pi_{r+1}(S_{2n+1})$ étend à toutes les valeurs de r l'invariant de Hopf défini par G. W. Whitehead.

H. Cartan (Paris).

Kodama, Yukihiro. On ANR for metric spaces. Sci. Rep. Tokyo Kyoiku Daigaku. Sect. A. 5 (1955), 96–98.

Let X be an ANR for metric spaces. The author proves that there exists a countable inverse system $\{N_i\}$ of simplicial complexes (not necessarily finite) such that

$$H_p(X)\approx\text{Inv Lim }H_p(N_i),\quad H^p(X)\approx\text{Dir Lim }H^p(N_i),$$
$$\pi_p(X)\approx\text{Inv Lim }\pi_p(N_i),\quad \pi^p(X)\approx\text{Dir Lim }\pi^p(N_i),$$
$$\dim(X)<2p-1.$$

Here H_p and H^p denote homology and cohomology

theory satisfying axioms 1, 2, and 5 of Eilenberg and Steenrod, and $\pi_p(X)$ and $\pi^p(X)$ are homotopy and cohomotopy group of X, and Inv Lim and Dir Lim mean inverse limit and direct limit respectively. Moreover, if X is locally compact, then there exists a closed mapping of Inv Lim N_i onto X. As a consequence of this result, the author indicates that any two homology theories satisfying the axioms 1–7 of Eilenberg and Steenrod coincide on any ANR for metric spaces. *S. T. Hu.*

Mardešić, Sibe. Un théorème de dualité concernant les groupes d'homologie de l'espace fonctionnel $S_m{}^X$. C. R. Acad. Sci. Paris **242** (1956), 2214–2217.

In this note, the author shows that the homology group $H^k(X, P)$ of a k-dimensional compactum X with coefficients in the group P of reals mod 1 and the group $H_X{}^{m-k}(S_m, J)$ introduced in an earlier note [same C. R. **242** (1956), 983–984; MR **17**, 993], form a dual pair in sense of Pontrjagin. As a consequence, the homology group $H^{m-k}(S_m{}^X, J)$ with integer coefficients of the mapping space $S_m{}^X$, $k<m$, is isomorphic to the cohomology group $H_k(X, J)$. *S. T. Hu* (Detroit, Mich.).

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★ **Aleksandrov, P. S. Topologičeskie teoremy dvoĭstvennosti. Čast' I. Zamknutye množestva. [Topological duality theorems. Part I. Closed sets.]** Trudy Mat. Inst. Steklov. no. 48. Izdat. Akad. Nauk SSSR, Moscow, 1955. 110 pp. 5.10 rubles.

The main goal of this paper is Pontryagin's duality theorem: Let M^n be a closed homology-n-manifold (triangulable) with $H^p(M^n)=H^{p+1}(M^n)=0$ for some coefficient group G; let A be a closed subset of M^n, and B its complement; then with $q=n-p-1$, $H^p(A)\approx H_q(B)$ [here H^r means Čech-cohomology, and H_r compact Vietoris-homology]. No mention of J. W. Alexander is made.

All the machinery is developed: Finite complexes, homology and cohomology, subdivision, consolidation, a chapter on Pontryagin group-duality and limit groups, coverings, Čech-homology and cohomology, compact Vietoris homology, manifolds, dual subdivisions, Poincaré duality. All proofs are given in detail, and as the author states, the paper can be read as an introduction to topology. Relative homology plays a very minor role; the boundary operator and the exact sequence of a pair are not mentioned explicitly. The usual proof of $H^p(A)\approx H^{p+1}(M, A)$ from the exact sequence appears here, in the case where A is a subcomplex and the groups are simplicial, in a different version: (1) the p-coboundaries in A are exactly the images of cocycles of M under the injection, so that $H^p(A)=Z^p(A)-i^*Z^p(M)$; (2) the $(p+1)$-cocycles of M mod A are exactly those cochains of M mod A which in M are coboundaries, so that $H^{p+1}(M, A)=j^{*-1}(B^{p+1}(M))-B^{p+1}(M,A)$; (3) with no restriction on the homology of M^n (and in fact without the manifold property), there is an isomorphism between $Z^p(A)-i^*Z^p(M)$ and $j^{*-1}(B^{p+1}(M))-B^{p+1}(M, A)$; point (3) is actually stated only for manifolds, interpreting the second group in the dual subdivision.

The second part wil bring the more difficult duality theorems of the author and Sitnikov for arbitrary subsets A. *H. Samelson* (Ann Arbor, Mich.).

Whitehead, J. H. C. Duality in topology. J. London Math. Soc. **31** (1956), 134–148.

This is an expository paper summarizing the results and historical development of various types of duality in topology. The author starts with familiar purely algebraic dualities and shows how the character theory for locally compact abelian groups leads to duality between homology and cohomology groups. Included is a description of the singular homology groups of a space X as a direct limit of homology groups of polyhedra mapped into X whereas the Čech cohomology groups of X are obtained as a direct limit of cohomology groups of polyhedra into which X is mapped.

There is then a complete survey of the (absolute and relative) duality in a manifold. Finally the duality in the homotopy theory in the stable range is discussed. The suspension category is defined, and a statement of duality in this category is given. The author concludes with a discussion of the concept of coconnectedness which is dual to the concept of connectedness of a space.

E. Spanier (Chicago, Ill.).

Fáry, István. Valeurs critiques et algèbres spectrales d'une application. Ann. of Math. (2) **63** (1956), 437–490.

This is a study of the homology properties of a continuous mapping in terms of critical sets of the mapping. The main theorem asserts that given a (sufficiently nice) continuous map $f:X\to Y$ there exists a spectral sequence whose E_2-term is described in terms of cohomology of subsets of Y relative to (generalized) local coefficients (the subsets being determined by the critical sets) and whose E_∞-term contains the graded group of a suitable filtration of the cohomology of X. The spaces considered are all locally compact and cohomology means cohomology with compact supports. It is also assumed that there is given a fixed commutative ring A, and all algebras will be algebras over A.

The author begins with concepts relating to sheaves (faisceaux). A sheaf $\mathfrak{A}$ on a space X is a function which assigns to every closed subset $E\subset X$ an algebra $\mathfrak{A}(E)$ and to every pair $F\subset E$ of closed subsets of X a map $\mathfrak{A}(E)\to\mathfrak{A}(F)$ such that the obvious transitivity condition is satisfied when $G\subset F\subset E$ and such that each $\mathfrak{A}(E)$ is isomorphic to the direct limit of $\mathfrak{A}(F)$ as F varies over closed neighborhoods of E. The cohomology of X relative to a sheaf $\mathfrak{A}$, denoted by $H(X, \mathfrak{A})$, is defined by a slight modification of the method of Leray [J. Math. Pures Appl. (9) **29** (1950), 1–139; MR **12**, 272].

There is then defined the relation of punctual isomorphism between sheaves on the same space X. Without defining this we state the important property that if $\mathfrak{A}$ and $\mathfrak{B}$ are sheaves on X which are punctually isomorphic then $H(X, \mathfrak{A})$ is isomorphic to $H(X, \mathfrak{B})$. There is also defined the concept of locally constant sheaf on a space X and, for locally connected spaces X, the concept of punctually constant sheaf (a sheaf which is punctually isomorphic to a locally constant sheaf). A punctually constant sheaf is a generalization of a local coefficient system; in fact, for connected, locally arcwise connected, locally simply connected spaces X the cohomology of X relative to a punctually constant sheaf is isomorphic to the cohomology of X relative to a bundle of coefficients in the sense of Steenrod [The Topology of Fibre Bundles, Princeton University Press, 1951; MR **12**, 522] and, therefore, can be regarded as classical.

Given a sheaf $\mathfrak{A}$ on X a point $x\in X$ is called a noncritical point of $\mathfrak{A}$ if there is a closed neighborhood V of x such that the restriction of $\mathfrak{A}$ to V is punctually isomorphic to a constant sheaf. Other points of X are called

critical points of $\mathfrak{A}$. Let X_1 denote the set of critical points of a sheaf $\mathfrak{A}$. Clearly X_1 is a closed subset of X. Let X_2 denote the subset of X_1 composed of critical points of $\mathfrak{A}|X_1$ (the restriction of $\mathfrak{A}$ to X_1). Continuing in this way there is defined a sequence

$$X=X_0\supset X_1\supset X_2\supset\cdots$$

in which X_n is the set of critical points of $\mathfrak{A}|X_{n-1}$ for $n\geq 1$. It is shown that if the sheaf $\mathfrak{A}$ has the property that $\bigcap_{q=0}^{\infty}X_q$ is empty, then there is a spectral sequence whose E_1-term is $\sum_{q\geq 0}H(X_q-X_{q+1},\mathfrak{A})$ and whose E_∞-term contains the graded group of $H(X,\mathfrak{A})$ defined by a suitable filtration of $H(X,\mathfrak{A})$. It is also proved that if X_q is locally connected, $\mathfrak{A}|X_q-X_{q+1}$ is punctually isomorphic to a locally constant sheaf $\mathfrak{A}_q$ so

$$H(X_q-X_{q+1},\mathfrak{A})\approx H(X_q-X_{q+1},\mathfrak{A}_q),$$

where the latter is a cohomology group relative to a local system. Therefore, if $\bigcap_{q=0}^{\infty}X_q$ is empty and each X_q is locally connected, the E_1-term of the spectral sequence involves classical cohomology groups (relative to local coefficient systems); so the spectral sequence is a bridge from such groups to the cohomology $H(X,\mathfrak{A})$ relative to the sheaf $\mathfrak{A}$.

Let $f:X\to Y$. The image sheaf $\mathfrak{S}$ of f is defined by $\mathfrak{S}(E)=H(f^{-1}E)$ where the latter denotes the cohomology algebra over A. A point $y\in Y$ is called a critical value of f if it is a critical point of the image sheaf $\mathfrak{S}$. It is shown that if Y is locally connected and $f(Y)=X$ then $y_0\in Y$ is a non-critical value of f if and only if there is a closed connected neighborhood V of y_0 such that for $y\in V$ the natural map $H(f^{-1}V)\to H(f^{-1}y)$ is onto and has kernel independent of y. f is said to have simple homology type if its critical sets

$$Y=Y_0\supset Y_1\supset Y_2\supset\cdots$$

are each locally connected and $\bigcap_{q=0}^{\infty}Y_q$ is empty. If f is a fibering, then Y_1 is empty; so f has simple homology type if Y is locally connected. If X and Y are locally finite simplicial complexes and $f:X\to Y$ is simplicial, then it is shown that f has simple homology type.

The principal result asserts that if $f:X\to Y$ has simple homology type and if

$$Y=Y_0\supset Y_1\supset Y_2\supset\cdots$$

denotes the sequence of successive critical values of f, there is a spectral sequence whose E_∞-term contains the graded algebra of $H(X)$ defined by a filtration and whose E_2-term is given by

$$E_2=\sum_{q\leq 0}H(Y_q-Y_{q+1},\mathfrak{S}_q),$$

where $\mathfrak{S}_q$ is a locally constant sheaf on Y_q-Y_{q+1} such that if $y\in Y_q-Y_{q+1}$ then $\mathfrak{S}_q(y)\approx\sum_{i\geq 0}H^i(f^{-1}y)$. It is further proved, in a theorem attributed to Leray, that the differential d_2 of E_2 is the sum of two differentials, one of which is a coboundary in an exact sequence and the other is similar to the second differential of the spectral sequence of a fibering. If f is a fibering, the spectral sequence of the main theorem is the classical spectral sequence of the fibering; so the main theorem can be regarded as a generalization of the classical spectral sequence of a fibering to fiberings with singularities.

No applications are given in the present paper. These are promised for a second paper which is to apply the main theorem to the cohomology theory of algebraic varieties. *E. Spanier* (Chicago, Ill.).

Hattori, Akira. On exact sequences of Hochschild and Serre. J. Math. Soc. Japan **7** (1955), 312–321.

Two exact sequences for the cohomology of group extensions which were originally established by using spectral sequences [Hochschild and Serre, Trans. Amer. Math. Soc. **74** (1953), 110–134; Ths. 2 and 3; MR **14**, 619] are obtained here by an elementary and direct procedure. Let G be a group, K a normal subgroup of G, A a G-module, $Z(G)$ the group algebra of G over the integers. The basic device for the author's procedure is a simultaneous dimension reduction for the cohomology of G, K, and G/K, using $\mathrm{Hom}_Z(Z(G),A)$ as a splitting module for G and K, and its K-fixed part as a splitting module for G/K. *G. P. Hochschild* (Berkeley, Calif.).

Adams, J. F.; and Hilton, P. J. On the chain algebra of a loop space. Comment. Math. Helv. **30** (1956), 305–330.

This paper is concerned with the following important problem: Given a topological space X, determine the structure of the homology ring $H_*(\Omega X)$, where ΩX denotes the space of loops in X (the multiplication in the homology ring is the Pontrjagin product). The main construction may be desccribed as follows. Let K be a CW-complex which has only one vertex and no 1-cells. For each integer $n>1$ let T_n denote the set of all cells of K of dimension n, let T denote the union of the sets T_n, and let A denote the free associative ring (with a unit) generated by the set T. Give the ring A a graded structure by assigning to each generator belonging to T_n the degree $(n-1)$. The authors prove that it is possible to define a differential operator $d:A\to A$ in the ring A which is homogeneous of degree -1, is an anti-derivation with respect to products, and such that the derived ring $H_*(A)$ is isomorphic to the homology ring $H_*(\Omega K)$ of the space of loops in K. The differential operator d is constructed stepwise on the generators of successively higher degrees.

The authors give several examples and applications of this construction. They also give a construction which (theoretically at least) makes possible the computation of the homomorphism $H_*(\Omega K_1)\to H_*(\Omega K_2)$ induced by a continuous map $K_1\to K_2$, where K_1 and K_2 are complexes of the type discussed in the preceding paragraph. Finally, they show how $H_*(\Omega(K_1\times K_2))$ may be computed if one knows a construction of the kind mentioned above for computing $H_*(\Omega K_1)$ and $H_*(\Omega K_2)$. *W. S. Massey.*

Kosiński, A. A note on labil points. Colloq. Math. **4** (1956), 11–12.

A point p of a (separable metric) space K is called a homotopically labil point (h.l.p.) if for each neighborhood U of p there is a deformation $f(x,t)$ of $\bar{U}$ into itself which is the identity on $\mathrm{Fr}(\bar{U})$ and p is not in $f(\bar{U},1)$. A neighborhood U of a point p is called star-shaped if $\bar{U}$ is an AR (absolute Retract) and $\mathrm{Fr}(\bar{U})$ is a retract of $\bar{U}-(p)$. The author generalizes a theorem of Noguchi [Kōdai Math. Sem. Rep. **1954**, 13–16; MR **16**, 60] to what he calls regular spaces; a regular space being defined as a space K every point of which has arbitrarily small star-shaped neighborhoods. The generalized theorem reads: A point p of a regular space K is h.l.p. in K if and only if boundaries of arbitrarily small star-shaped neighborhoods of p are absolute retracts. *Haskell Cohen.*

Sugawara, Masahiro. On fibres of fibre spaces whose total space is contractible. Math. J. Okayama Univ. **5** (1956), 127–131.

Let (E,F,B,p) be a fibre-space in the sense of Serre

and let E be a CW-complex, F a locally finite CW-complex. The author shows that if E is contractible to the base point $x_0 \in F$, rel x_0, then F may be given an H-structure with x_0 as unit element; the H-structure is homotopy associative with (two-sided) homotopy inverse.

The argument proceeds by setting up an H-structure in F by means of a given contraction of E and then defining a homomorphism from F to the space of loops on B which is a singular homotopy equivalence. *P. J. Hilton.*

Milnor, John. Construction of universal bundles. II. Ann. of Math. (2) **63** (1956), 430–436.

The author starts from a given topological group G, rather than a given base as in part I. It is shown that for any given topological group G (not necessarily compact Lie, as hitherto known) there is an n-universal bundle ($n \leq \infty$) E_n having G as group. E_n is in fact the join of $(n+1)$ copies of G with suitable topology. Denoting the base space by B_n the author then investigates the homology of B_∞. It is first shown that there is an embedding $B_0 \subset B_1 \subset B_2 \subset \cdots \subset B_\infty$ and that $H_k(B_n, B_{n-1})$ is isomorphic to the reduced group $H_{k-1}(E_{n-1})$, n, $k>0$; in case G is torsion-free, the latter group is expressly calculated. Using the spatial inclusion, a spectral sequence $\{E_{p,q}{}^r\}$ is constructed with $E_{p,q}{}^1 = H_{p+q}(B_p, B_{p-1})$ whose limit term is the graded homology of B_∞. In the last section, interconnections with part I are given. Assume that G is a countable CW complex with cellular group operations. Then (1): B_∞, E_∞ can be chosen to be countable CW complexes. (2): The given G is the group of two ∞-universal bundles having countable CW complex bases B, B' respectively, if and only if B, B' belong to the same homotopy type (3): Given a countable connected CW complex B, it will serve as base of two ∞-universal bundles with CW groups G, G' if and only if there is a third such group G'' which can be mapped homomorphically into each of G, G' by homotopy equivalences.
J. Dugundji (Los Angeles, Calif.).

Wu, Wen-Tsün. On Pontrjagin classes. II. Acta Math. Sinica **4** (1954), 171–199. (Chinese. English summary)

This paper contains the detailed proofs of the results sketched in an earlier note of the author [Colloque de Topologie de Strasbourg, 1951, no. IX, Bibliothèque Nat. et Univ. de Strasbourg, 1952; MR **14**, 491]. It is proved that for a closed differentiable manifold, for any odd prime p, certain combinations by means of cup products of the Pontrjagin classes, reduced mod p, with respect to any given differential structure of the manifold are in reality independent of that structure and therefore are topological invariants of the manifold. For $p=3$, it turns out that the mod 3 Pontrjagin classes are themselves topological invariants of the manifold. The proofs are based on the so-called "diagonal method", obtaining thus a connection between the Pontrjagin classes and the Steenrod powers, of which the latter ones, in thec ase of a differentiable manifold, are of purely topological character. *S. T. Hu* (Detroit, Mich.).

Wu, Wen-Tsün. On Pontrjagin classes. IV. Acta Math. Sinica **5** (1955), 37–63. (Chinese. English summary)

This paper contains a second proof that the Pontrjagin classes reduced mod 3 of a closed differentiable manifold are topological invariants. The proof is different from the preceding one given in the paper reviewed above. In fact, Steenrod powers do not come explicitly in the considerations. *S. T. Hu* (Detroit, Mich.).

Wu, Wen-Tsün. On Pontrjagin classes. V. Acta Math. Sinica **5** (1955), 401–410. (Chinese. English summary)

In this paper, the author gives an explicit formula for the Q-classes introduced in the paper reviewed second above. As a corollary, the following result is deduced: In an oriented closed differentiable manifold M of dimension $2(p-1)$ or $2p-1$, $Q_p{}^{2(p-1)}$ vanishes. In particular, $P_3{}^4=0$ in an oriented closed differentiable manifold of dimension 4 or 5. *S. T. Hu* (Detroit, Mich.).

Vaccaro, Michelangelo. Sulla caratteristica dei complessi simpliciali χ-omogenei. Ann. Mat. Pura Appl. (4) **41** (1956), 1–20.

A finite simplicial complex is called χ-homogeneous in the simplex e^r if $\chi \mathrm{St}(e^r) = (-1)^n$, where χ is the Poincaré characteristic. The author studies the consequences of various hypotheses about χ-homogeneity. Typical hypotheses are (1) n is odd and K^n is χ-homogeneous in r-simplexes where $n-r$ is odd; (2) n is even and K^n is χ-homogeneous in even dimensional simplexes. In both cases, the author obtains complete sets of linear relations which must be satisfied by $\alpha_1, \cdots, \alpha_n$ (α_k being the number of k-simplexes). These relations lead to special formulas for $\chi(K^n)$. For example in case (2), $\chi K^2 = \alpha^0 - \alpha^2/2$, $\chi K^4 = \alpha^0 - \alpha^2/2 + \alpha^4$ etc. It is shown that $\chi K^n = 0$ when n is odd and K^n is χ-homogeneous in even dimensional simplexes. *P. A. Smith* (New York, N.Y.).

Bourgin, D. G. Un indice dei punti uniti. I, II. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 435–440; **20** (1956), 43–48.

In these two notes the author lays the foundations for a fixed-point-index theory for transformations f: $X \to X$ where X is a compact Hausdorff space, assumed to be an ANR. Using Čech homology theory over rational coefficients, the author exploits the fact that the homology theory of such an X, and its influence under f, can be transferred to a finite approximating polytope where, of course, an index theory is readily available. The principal results are 1) the Lefschetz number of f equals the Lefschetz number of the transformation induced in the nerve of any sufficiently fine covering; 2) if G is an open subset of X whose boundary contains no fixed points, the index of the transformation induced in a sufficiently fine nerve over a certain open subset of the nerve, determined by G, is independent of the nerve chosen. *P. A. Smith.*

Weier, Josef. Über Klassifizierung homotoper Abbildungen. Monatsh. Math. **60** (1956), 157–167.

Let P be a compact locally euclidean manifold and let g^0, g^1 be two null-homotopic fixed-point free mappings of P into itself. It is shown that there exists a homotopy g^t from g^0 to g^1 such that the set or points (p, t) of $P \times I$ for which $g^t(p) = p$ is empty or else consists of a finite number of disjoint simple closed curves. *P. A. Smith.*

Sitnikov, K. A. Combinatorial topology of nonclosed sets. II. Dimension. Mat. Sb. N.S. **37(79)** (1955), 385–434. (Russian)

This is a detailed presentation of results announced and described in several notes [Dokl. Akad. Nauk SSSR (N.S.) **82** (1952), 845–848; **83** (1952), 31–34; **88** (1953), 21–24; **94** (1954), 1007–1010; MR **13**, 860; **14**, 894; **15**,

978]. Ch. I brings several characterizations of dimension of a set $A \subset E^n$: (1) by essential maps into a simplex (after P. S. Aleksandrov), in a strengthened form (existence of essential maps which can be extended to a neighborhood of A); (2) by damped maps: The intersection $\Gamma \cap A$ (Γ open) can be deformed into an r-dimensional polyhedron by a map which "goes to 0" at the boundary of $\dot{\Gamma}$; (3) by existence of non-trivial cocycles in some relatively closed subset H of A modulo the boundary $\dot{H}$ of H. The example of ref. 3 above is discussed in detail. Ch. II brings a proof of the theorem on obstructions: If $A \subset E^n$ has dimension r, then for some open sphere U^n there is a non-bounding $(n-r-1)$-cycle (in the author's sense, integral coefficients) in $U^n - A$; for no open sphere U^n is there a non-bounding q-cycle ($q<n-r-1$) in $U^n - A$. Ch. III describes the example of ref. 4 above of a 2-dimensional subset of E^3, which does not separate any open subset of E^3 Ch. IV brings the proof of the deformation theorem of ref. 1 above: Let A be a subset of a closed manifold M, and let z be a Sitnikov-cycle on a compact subset ϕ of $M-A$; let g_t and h_t be deformations of A and ϕ, such that $g_t(A) \cap h_t(\phi) = \emptyset$ for all t; suppose $h_1(z) \sim 0$ in $M - g_1(A)$; then $z \sim 0$ in $M-A$. Several applications are made, e.g.: If $\dim \bar{A} = \dim A = r$ ($\bar{A}$=closure of A), then A does not admit arbitrarily small displacements into polyhedra of dimension $<r$. *H. Samelson.*

See also: Borel, p. 1108; Patterson, p. 1115; Kinohara, p. 1047; Wallace, p. 1131.

GEOMETRY

Have

★ **Tietz, H. Geometrie.** Handbuch der Physik. Bd. II., pp. 117–197. Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. DM 88.00.

This part of the second volume of the Handbuch der Physik gives a good account of those geometric ideas that are most important for physicists. Here is a survey. A. Elementary analytical geometry: I. Vector calculus; II. n-dimensional geometry and matrices, orthogonal transformations; III. projective geometry, elliptic, hyperbolic and parabolic geometry. B. Elementary differential geometry: I. Curves; II. surfaces. C. Elementary field theory, Gauss, Stokes, Poisson. D. "Higher" geometry: I. Ricci calculus, parallelism, the V_n and the W_n; II. spinors; III. contact transformations. There are some references and a literature list.

J. A. Schouten (Epe).

Dingler, Hugo. Geometrie und Wirklichkeit. Dialectica 9 (1955), 341–362.

Tallini, Giuseppe. Su una estensione del teorema di Desargues. Boll. Un. Mat. Ital. (3) 11 (1956), 46–48.

Verf. bemerkt zu der Arbeit von P. O. Bell [Proc. Amer. Math. Soc. 6 (1955), 675–681; MR 17, 292], dass die beiden dort bewiesenen Sätze, die das Theorem von Desargues und seine Umkehrung auf Räume beliebiger Dimension übertragen, sich bereits bei B. Segre [Lezioni di geometria moderna, vol. I, Zanichelli, Bologna, 1948, S. 166; MR 10, 729] finden. Verf. beweist ferner auf synthetischem Wege zwei analoge, für beliebige graphische irreduzible Räume gültige Sätze, aus denen die Sätze von Bell folgen.

R. Moufang (Frankfurt am Main).

Wunderlich, W. Formeln und Rechenbehelfe zur Abwicklung des Kegels 2. Ordnung. Österreich Ing.-Arch. 10 (1956), 107–114.

This paper is concerned with the evaluation of the angle between the cuts when a right-elliptic cone is developed on the plane by a cut down generator. This angle, 4ω, is expressed in the form $\omega(\alpha, \beta)$, where 2α and 2β are the angles subtended at the vertex by the major and minor axes of the ellipse. The function $\omega(\alpha, \beta)$ is derived explicitly in terms of complete and incomplete elliptic integrals, and a double entry table with two sets of contour graphs is given. An approximate formula is then derived, from which an alignment nomogram and special slide-rule is evolved.

J. G. L. Michel (Teddington).

Have

★ **Russell, Bertrand A. W. An essay on the foundations of geometry.** Dover Publications, Inc., New York, 1956. xx+201 pp. Paperbound: $1.50; clothbound: $3.25.

Photo-offset reprint of the first edition [Cambridge, 1897]. There is a new foreword by Morris Kline.

Rossier, Paul. Les axiomes de la géométrie multidimensionelle. Arch. Sci. Soc. Phys. Hist. Nat. Genève 8 (1955), 449–456.

This is an expository article, along the lines of the first two chapters of D. M. Y. Sommerville [An introduction to the geometry of n dimensions, Methuen, London, 1929]. To deduce the consistency of projective 4-space from the consistency of the affine plane, the author represents the ∞^4 points of the former by the ∞^4 parabolas of the latter. The equation for the general parabola is stated incorrectly, but that does not affect the argument.

H. S. M. Coxeter (Toronto, Ont.).

Arvesen, Ole Peder. Zur Frage der Anschaulichkeit der Bilder aus einem vierdimensionalem Raume. Norske Vid. Selsk. Forh., Trondheim 28 (1955), 166–170 (1956).

Bager, Anders. Inscribed and escribed circles and sums of powers. Nordisk Mat. Tidskr. 4 (1956), 30–35, 64. (Danish. English summary)

The author examines in some detail the relations connecting the edges in a triangle, its heights, area, circumference and radii of the circumscribed and the four inscribed circles, in particular, the expressions for the power sums of the latter. *O. Ore.*

Froda, Alexandru. Sur les triangles rationnels. Com. Acad. R. P. Romîne 5 (1955), 1695–1701. (Romanian. Russian and French summaries)

The classical relations

$$(*)\qquad a/(m^2+n^2) = b/(m^2-n^2) = c/2mn = \varrho$$

(ϱ rational, $m>n>0$, rational integers), necessary and sufficient, in order to have a right triangle with rational sides a, b, c is generalized for arbitrary triangles. The conditions

$$(**)\qquad a/((m^2+n^2)q-mnp) = \varepsilon b/(m^2-n^2)q = \varepsilon c/n(2mq-np) = \varrho$$

($\varrho>0$, rational, m, n coprime integers, $n>0$, p, q coprime

integers, $q>0$, $\varepsilon=\pm1$) are necessary and sufficient for the existence of a non-degenerate triangle of rational sides a, b, c. In the particular case $p=0$, (**) reduce to (*).
E. Grosswald (Philadelphia, Pa.).

Have

★ **[Anspach, Pierre A. L.] Aperçu de la théorie des polygones réguliers (suite).** Bruxelles, 1956. pp. 93–192.

To the published review of the first part of this book [1955; MR **17**, 397] it may suffice to add here the author's table of contents of this second part. 1. Remarkable points of the heptal triangle. 2. Six conics circumscribed about a heptal triangle. 3. "Exchanges" between two triangles similar to the heptal triangle. 4. A group of three triangles which may be considered to be satellites of the heptal triangle.

It should perhaps be mentioned that the heptal triangle is the object of a note by V. Thébault [Mathesis **65** (1956), 106–109], and that the book under review is cited there among the bibliographical references. *N. A. Court.*

Fuhrmann, Arved. Klassen ähnlicher Matrizen als verallgemeinerte Doppelverhältnisse. Math. Z. **62** (1955), 211–240.

A rectangular matrix $\mathfrak{M}$ of n rows with elements in a field (not assumed commutative) is regarded as a coordinate matrix for the space spanned by its columns and the space determines a class of right equivalent matrices. If ${}^0\mathfrak{M}$ is any matrix with rows spanning the dual space, ${}^0\mathfrak{M}_1\cdot\mathfrak{M}_2$ will have an inverse for ${}^0\mathfrak{M}_1$ and $\mathfrak{M}_2$ of minimal size if and only if the spaces determined by $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are complementary (abbreviated: $1\oplus2$).

The square matrix

$$\mathfrak{A}=({}^0\mathfrak{M}_1\cdot\mathfrak{M}_4)^{-1}\cdot({}^0\mathfrak{M}_1\cdot\mathfrak{M}_3)\cdot({}^0\mathfrak{M}_2\cdot\mathfrak{M}_3)^{-1}\cdot({}^0\mathfrak{M}_2\cdot\mathfrak{M}_4)$$

is well defined if and only if $1\oplus4$ and $2\oplus3$. The spaces then determine A to within an arbitrary similarity and this similarity class is the cross ratio (1, 2; 3, 4), generalizing the classical special cases. If $1\oplus2$, $1\oplus3$, $1\oplus4$ and $2\oplus3$ are all assumed, then (1, 2; 3, 4), (2, 1; 4, 3), (3, 4; 1, 2) and (4, 3; 2, 1) all exist and are equal, if (1, 3; 2, 4) exists it is the class of $\mathfrak{A}^{-1}$; similar results analogous to classical ones are obtained for all the permutations. Using normalized coordinate matrices, it is proved that if $2\oplus3$, $1\oplus3$, $1\oplus4$ and 1 and 2 span the whole space (but may intersect), then these position restrictions with (1, 2; 3, 4) and (4, 3; 2, 1) are a complete invariant system of the configuration under collineations.

The relation of the fourth space to the Segre manifold determined by the other three when $1\oplus2$, $1\oplus3$ and $2\oplus3$ is studied for commutative fields of zero characteristic. In a final section, it is proved that an arbitrary n by n matrix $\mathfrak{A}$ is similar to a direct sum of a matrix in Jordan normal form with its eigenvalues in the center $\mathbf{Z}$ of the coefficient field and a matrix determined to within similarity and having no eigenvalues in $\mathbf{Z}$. *W. Givens.*

Artzy, Rafael. Self-dual configurations and their Levi graphs. Proc. Amer. Math. Soc. **7** (1956), 299–303.

Does every self-dual configuration admit an involutory reciprocity? In other words, is it always possible to assign corresponding symbols X to the points and x to the lines (or planes) in such a way that a is incident with B whenever A is incident with b? Assuming an affirmative answer, the author defines, for a given configuration and a given involutory reciprocity, a "reduced Levi graph" having a node X for each dual pair (X, x) and a branch AB for each incidenceor . Fexample, the Pappus configuration 9_3 with any one of Kommerell's 18 reciprocities [see Coxeter, Bull. Amer. Math. Soc. **56** (1950), 413–455, p. 434; MR **12**, 350] yields a hexagon having loops at alternate vertices while the remaining three vertices are joined to respective vertices of a triangle. This illustrates the fact that a regular configuration n_d yields a graph having n nodes and $dn/2$ branches, each loop counting as half a branch. When n and d are both odd, there must be at least one loop; therefore the configuration must then have at least one point incident with its dually corresponding line (or plane). On the other hand, the Desargues configuration 10_3 with von Staudt's polarity [ibid., p. 436] yields the Petersen graph, without loops. The various self-dual specializations of the Desargues configuration [see G. Pickert, Projektive Ebenen, Springer, Berlin, 1955; MR **17**, 399] are derived by adding loops at appropriate nodes of the graph. Another graph without loops is formed by the 12 vertices and 18 edges of the truncated tetrahedron; this arises from the reviewer's configuration 12_3 [Coxeter, loc. cit., p. 438]. A graph having a loop at every node arises when a three-dimensional configuration of points and planes admits a null polarity; e.g., for Möbius's 8_4 consisting of two mutually inscribed tetrahedra [ibid., p. 444] the graph is formed by the vertices and edges of a cube with a loop at each vertex.
H. S. M. Coxeter (Toronto, Ont.).

Gulyaeva, L. A. Some properties of a system of conic sections of space doubly intersecting some fixed conic section. Moskov. Oblast. Pedagog. Inst. Uč. Zap. Trudy Kafedr Mat. **20** (1954), 145–172. (Russian)

The main object of this paper is to study synthetically the properties of non-linear systems of conics $[y^2]$ lying on the system of quadric surfaces $\bar{S}_4$ passing through a given conic y^2.

The method employed is very closely analogous to the one the author used in an earlier paper [Kirov. Gos. Ped. Inst. Uč. Zap. **1953**, no. 7, 29–47; MR **17**, 885] and consists, in the present case, in establishing a one-to-one projective correspondence between the quadrics $\bar{S}_4$ in ordinary space and the points of the four dimensional projective space S_4.

The plan and the method of exposition also follow, skillfully and with discernment, the pattern set by the paper referred to.

When the conic y^2 coincides with the imaginary circle at infinity, the systems $\bar{S}_4$ and $[y^2]$ become, respectively, the totality of spheres and circles in three-space. The author is thus led to consider, among other things, some configurations in three-space which have already been discussed by others, adding for them a synthetic proof of her own. *N. A. Court* (Norman, Okla.).

Lombardo-Radice, Lucio. A proposito di un teorema sui piani finiti sopra un quasicorpo. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 599–601.

In a recent paper unavailable to the reviewer [same Rend. (8) **18** (1955), 154–161; MR **17**, 886] the author has proved the following theorem: Let π be a finite projective plane admitting all homologies with respect to a given axis and a given center not on this axis. If Fano's configuration 7_3 is universal in π, then π is desarguesian. The present note fills a gap which the author asserts that he has found in his proof of the stated result.
R. L. San Soucie (Eugene, Ore.).

Karzel, Helmut. Über eine Anordnungsbeziehung am Dreieck. Math. Z. **64** (1956), 131–137.

An order function $h(\alpha)$ in a projective (incidence) plane is a function, defined for all lines h and all points α, which is 0 when h, α are incident, and ± 1 otherwise [see Sperner, Math. Ann. **121** (1949), 107–130; S-B. Heidelberger Akad. Wiss. Math.-Nat. Kl. **1949**, 413–448; MR 12, 43; **16**, 278]. Three distinct points α, β, γ on a line k satisfy the relation $[\alpha, \beta|\gamma]$ if and only if for any two lines g, h $(\neq k)$ through γ, $g(\alpha)g(\beta)h(\alpha)h(\beta)=1$. $[a, b|c]$ is defined dually. If any distinct collinear α, β, γ satisfy $[\alpha, \beta|\gamma]$ then the order function is said to satisfy the line relation. The line relation implies the triangle relation: if α, β, γ are the vertices of a triangle, and a, b, c any lines whose intersections (a, b), (b, c), (c, a) are incident with the sides (α, β), (β, γ), (γ, α) of the triangle without coinciding with one of α, β, γ, then

$$a(\beta)a(\gamma)b(\gamma)b(\alpha)c(\alpha)c(\beta)=1.$$

The author proves the following converse: for any order function $h(\alpha)$ in a projective plane in which the minor theorem of Desargues holds and in which every line contains at least 4 points, the triangle relation implies the line relation provided $[\alpha, \beta|\gamma]$ holds for at least one triplet of points or $[a, b|c]$ holds for at least one triplet of lines.

F. A. Behrend (Melbourne).

Järnefelt, G.; und Qvist, Bertil. Die Isomorphie eines elementargeometrischen und eines Galois-Gitterpunktemodells. Ann. Acad. Sci. Fenn. Ser. A. I. no. **201** (1955), 11 pp.

A finite field can be pseudo-ordered by the convention that "$a>b$" means "$a-b$ is a nonzero square". This ordering is asymmetric if -1 is not a square, but it cannot be transitive. However, it may be transitive over a large subset of the field. For fixed integers k, m, and n, consider the geometry consisting of the points $(\mu m/kn, \nu m/kn)$, $\mu, \nu=0, \pm 1, \pm 2, \cdots, \pm k$, and those lines which meet at least two of these points. The authors give detailed proofs that, provided the prime p is suitably chosen, the properties (including order) of this geometry remain valid if the coordinates are regarded as lying in the pseudo-ordered field with p elements.

A. M. Gleason.

Palaj, Cyril. L'invariant Θ_{n+1} comme un invariant simultané fondamental d'une jusqu'à $n+1$ hyperquadriques dans l'espace à n dimensions. Czechoslovak Math. J. **5(80)** (1955), 345–354. (Russian summary)

The present paper extends to n dimensions and $n+1$ hyperquadrics the theory of simultaneous invariants of conics and quadrics in 2- and 3-space which the author has developed in an earlier paper [Časopis Pěst. Mat. Fys. **75** (1950), 159–177; MR **13**, 153]. Let $k_\nu=\sum_{i,j=1}^{n+1} a_{\nu ij}x_ix_j=0$ $(\nu=1, \cdots, n+1)$ be the equation of $n+1$ hyperquadrics in projective n-space. The invariant in question is represented by the determinant of a cubic matrix whose νth layer is the discriminant matrix of the νth quadric k_ν, i.e. $\theta_{n+1}=|a_{\nu ij}|$. A complicated geometrical interpretation is given of the condition $\theta_{n+1}=0$. Only readers having a detailed knowledge of the not widely known theory of cubic determinants will be able to follow the author's arguments.

H. Schwerdtfeger (Melbourne).

See also: von Schilling, p. 1159.

Convex Domains, Extremal Problems, Integral Geometry, Metric Geometry

Have

★ **Blaschke, Wilhelm. Kreis und Kugel.** 2te Aufl. Walter de Gruyter & Co., Berlin, 1956. viii+167 pp. DM 18.60.

With the exception of a few very minor interpolations this is a photostatic reproduction of the 1916 edition [Veit, Leipzig]. Only the preface and the last part (pp. 162–164) have been rewritten. Reading the book is still a pleasant experience; however, reprinting the book without any pretense of having brought it up to date would have been preferable. As it is, both really recent work and work which was new in 1916 are referred to as recent (p. 82). Topics which have long since been studied are represented as unexplored (e.g. convex sets in Hilbert space, p. 85). Poincaré's assertion that there are three closed geodesics on a convex surface is reported, but neither M. Morse nor Lusternik-Schnirlman are mentioned. Also, there are inaccuracies in quotations of new results, e.g. the rigidity of general closed convex surfaces is ascribed to A. D. Alexandrov instead of Pogorelov; H. Lewy's contribution to Weyl's Problem is not mentioned etc. *H. Busemann* (Los Angeles, Calif.).

Sobczyk, Andrew. Symmetrical types of convex regions. Math. Mag. **29** (1956), 175–192.

A region C in E_n which is invariant under an involution is said to have involutory symmetry; if it is generated by rotation about a subspace, it is said to possess rotational symmetry. The paper deals with the possible types and interrelations of both kinds of symmetry. Some facts about possible regions which are invariants under various groups of transformations are considered; it is shown, for instance, that for any bounded convex or starlike region C with the point 0 in its interior, any possible linear transformation T under which C may be invariant is such that there exists a choice of basis vectors for E_n with respect to which T appears as an orthogonal transformation.

L. A. Santaló (Buenos Aires).

Have

★ **Vincensini, Paul. Sur l'application d'une méthode géométrique à l'étude de certains ensembles de corps convexes.** Colloque sur les questions de réalité en géométrie, Liège, 1955, pp. 77–94. Georges Thone, Liège; Masson & Cie, Paris, 1956. 250 fr. belges; 1900 fr. français.

The convex bodies [c. b.'s] under consideration lie in euclidean n-space and are bounded by hypersurfaces which have continuous tangent-hyperplanes and finite non-vanishing principal curvatures everywhere. The method alluded to consists of (i) constructing the central symmetrization $\mathfrak{K}^*$ or rather the vector body $2\mathfrak{K}^*$ of a c.b. $\mathfrak{K}$ and (ii) of extending the linear pencil $\mathfrak{K}_\theta=(1-\theta)\mathfrak{K}_0+\theta\mathfrak{K}_1$ connecting two c.b.'s $\mathfrak{K}_0$ and $\mathfrak{K}_1$ $[0\leqq\theta\leqq 1$; cf. Bonnesen-Fenchel, Theorie der konvexen Körper, pp. 51, 73. Springer, Berlin, 1934]. This extension is achieved e.g. beyond $\mathfrak{K}_1$ if $\mathfrak{K}_\theta$ remains convex for some $\theta>1$. After some introductory remarks, the author proves: Given $\mathfrak{K}$ and $\mathfrak{K}_1=\mathfrak{K}_1^*$. Then there is a $\mathfrak{K}_0$ and a real number ν such that $\mathfrak{K}_0^*=\mathfrak{K}_1$ and $\mathfrak{K}=\mathfrak{K}^*+\nu(\mathfrak{K}_0-\mathfrak{K}_0^*)$ [p. 88, formula (3). Let H and H_1 denote the support functions of $\mathfrak{K}$ and $\mathfrak{K}_1$ respectively. By the author's assumptions, the boundary of $\mathfrak{K}_1$ contains no straight line segments. Hence $H_0(u)=H_1(u)+(2\nu)^{-1}(H(u)-H(-u))$ is the support function of a c.b. $\mathfrak{K}_0$ if $|\nu|$ is large enough. This $\mathfrak{K}_0$ has the required properties]. The second last section observes

that the sums of the radii of principal curvature in corresponding pairs of opposite points of $\mathfrak{K}$ and $\mathfrak{K}^*$ add up to the same number. Finally, a new proof of the author's generalization of Helly's Theorem is given [Bull. Soc. Math. France **67** (1939), 115–119; MR **2**, 11].

P. Scherk (Kingston, Ont.).

Ohmann, D. Ungleichungen zwischen den Quermassintegralen beschränkter Punktmengen. III. Math. Ann. **130** (1956), 386–393.

The author has conjectured that, for three indices $n>r>p\geqq 0$, a suitable generalization of an inequality, between pth "cross-section" integrals of the theory of Convex Sets, holds at least for all closed sets. The first two notes [Math. Ann. **124** (1952), 265–276; **127** (1954), 1–7; MR **13**, 864; **15**, 738] dealt with the special cases $r=0$ and $p=n-1$. In this third note, the author observes that his conjecture implies that there are no further independent inequalities between the cross-section integrals and he completes the discussion of the case $r=0$ (for arbitrary sets) by treating the case of equality.

L. C. Young (Madison, Wis.).

Santaló, L. A. On the distribution of sizes of particles contained in a body given the distribution in its sections or projections. Trabajos Estadist. **6** (1955), 181–196. (Spanish. English summary)

Let Q be a convex body containing a large number of convex particles: it is desired to estimate the distribution of their sizes from a study of sample sections of Q if Q is opaque, or of sample projections if Q is transparent. Under suitable assumptions about random placing in Q of the particles it suffices to consider distributions based on the measures customary in integral geometry. Special cases of the resulting problem about sections, mostly for spherical particles, were treated by S. D. Wicksell [Biometrika **17** (1925), 84–99], W. P. Reid [J. Math. Phys. **34** (1955), 95–102; MR **16**, 1128], and others. The author considers the case in which all the particles have the same shape as a fixed convex body K, so that each of them is congruent to λK for some positive λ. Let $F(\lambda)d\lambda$ be the limit (for large Q) of the number per unit volume of Q of particles with "size" in $(\lambda, \lambda+d\lambda)$, and let $f(\sigma)d\sigma$ be that of the number, per unit area of a plane section of Q, of particle sections with area in $(\sigma, \sigma+d\sigma)$. His first problem is then to determine $F(\lambda)$ when $f(\sigma)$ and K are known. Denoting by V, M, and σ_m the volume, integral of the mean curvature, and greatest sectional area of K, he shows that the formula

$$F(\lambda)=-\frac{4\sigma_m s^p \sin p\pi}{pM}\int_s^\infty \frac{f'(\sigma)d\sigma}{(\sigma-s)^p},$$

where $s=\sigma_m\lambda^2$ and $p+1=M\sigma_m/(2\pi V)$, holds exactly when K is a sphere (in which case $p=\frac{1}{2}$ and the result was found by Wicksell [loc. cit.]) and approximately when K is nearly spherical. The approximation consists in the replacement of the kernel of a certain integral equation for $F(\lambda)$ by a kernel of Abel's type. [Remark. The form of the Abel kernel will not always be well suited for fitting the true kernel. In particular, the former is unbounded while the latter may be continuous, as can be verified by explicit calculation if K is a spheroid.]

In the author's second problem linear sections and chord lengths replace plane sections and section areas, and he applies a similar analysis. In particular, when K is a sphere he finds that $\pi F(\lambda)$ is equal to $-(d/d\lambda)\{\lambda^{-1}f(2\lambda)\}$ in terms of the new $f(\sigma)$. A third problem treated on the same lines deals with the lengths of plane sections of disc-shaped particles, and his remaining problems are concerned with projections onto a plane of disc-shaped or needle-shaped particles. Explicit general solutions are found for both projection problems.

H. P. Mulholland (Birmingham).

Nasu, Yasuo. On asymptotic conjugate points. Tôhoku Math. J. (2) **7** (1955), 157–165.

The concepts introduced by the reviewer in Trans. Amer. Math. Soc. **56** (1944), 200–274 [MR **6**, 97] are used. The space E is a, not necessarily symmetric, finitely compact (but in the present case not compact) metric space in which locally unique geodesics or extremals exist. If $x(\tau)$, $\tau\geqq 0$, represents a ray R, i.e. $x(\tau_1)$, $x(\tau_2)=\tau_2-\tau_1$ for $\tau_2\geqq\tau_1$, then a co-ray C from an arbitrary point q to R is a limit of a sequence of oriented segments from q_n to $x(\tau_n)$, where $q_n\to q$ and $\tau_n\to\infty$. In general C is not unique, but the co-ray to R from $x\in C-q$ is unique and a sub-ray of C. The union of all co-rays to R containing a given co-ray is called an asymptote A to R, and A is either a straight line or possesses an initial point a, which the author calls an asymptotic conjugate point to R. The paper studies the set $K(R)$ of all asymptotic conjugate points to R. If $K(R)$ contains an isolated point a, then $K(R)=a$ and the asymptotes to R with initial point a cover R (except for a) simply. The limit spheres with R as central ray coincide with the spheres $ax=$const. An example shows that $K(R)$ need not be closed. If E possesses the property of invariance and the co-ray to R from any point is unique then $K(R)$ is closed. If, in addition, E is a two-dimensional Finsler space of class C^4, then $K(R)$ is empty. *H. Busemann* (Los Angeles, Calif.).

See also: Ghika, p. 1107; Kramer, p. 1116; Pettis, p. 1111.

Differential Geometry

Şemin, Ferruh. Généralisation de quelques formules relatives à la théorie des surfaces. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **20** (1955), 173–180 (1956). (Turkish summary)

L'auteur attache à chaque point d'une surface réelle un trièdre birectangle dont les deux premiers vecteurs unitaires sont tangents aux lignes coordonnées, le troisième étant normal à la surface. Par l'emploi des formes extérieures, il obtient les formules de la théorie du repère mobile adaptées au cas envisagé. Il applique ensuite ces formules à la détermination, sous des formes invariantes nouvelles, des expressions de la courbure normale et de la torsion géodésique d'une courbe gauche quelconque de la surface considérée, de la courbure géodésique d'une telle courbe, et des formules de Codazzi-Gauss de la théorie générale des surfaces.

Il termine par une application relative à la caractérisation des surfaces à courbure totale constante par le fait que les lignes asymptotiques y déterminent un réseau de Tchebychef. *P. Vincensini* (Marseille).

Tryuk, K. On B-curvatures of curves on surfaces of the Euclidean space. Ann. Polon. Math. **2** (1955), 14–28.

The B-curvatures of a curve lying in a subspace of an affinely-connected space were defined by S. Gołąb [Trudy Sem. Vektor. Tenzor. Anal. **4** (1937), 360–362]. This paper treats these curvatures for curves lying on a

surface in Euclidean 3-space. Curves which are B-straight, B-plane, and B-skew respectively are given geometric characterizations. The B-curvatures are computed in terms of the Euclidean curvatures of the curve relative to the surface, and it is shown that the normalization used in the theory of B-curvatures is not the same as that in the Euclidean theory. A special study is made of the B-curvatures of curves lying on developable surfaces. *C. B. Allendoerfer* (Seattle, Wash.).

Vyčichlo, F. On pairs of surfaces having certain common differential invariants. Mat.-Fyz. Časopis. Slovensk. Akad. Vied **6** (1956), 21–30. (Czech. Russian summary)

Etant donné un couple des surfaces ${}^{\alpha}x^i={}^{\alpha}f^i(u, v)$ ($\alpha=1, 2$; $i=1, 2, 3$), on peut définir les invariants différentiels bien connus

$$
{}^1n=\frac{\partial^1 f}{\partial u}\times\frac{\partial^1 f}{\partial v},\quad {}^2n=\frac{\partial^2 f}{\partial u}\times\frac{\partial^2 f}{\partial v},
$$

$$
(1)\qquad b=\frac{\partial^1 f}{\partial u}\times\frac{\partial^2 f}{\partial v}-\frac{\partial^1 f}{\partial v}\times\frac{\partial^2 f}{\partial u},
$$

$$
s=\frac{\partial^1 f}{\partial u}\cdot\frac{\partial^2 f}{\partial v}-\frac{\partial^1 f}{\partial v}\cdot\frac{\partial^2 f}{\partial u},
$$

où $\partial^{\alpha}f/\partial u$, $\partial^{\alpha}f/\partial v$ sont les vecteurs tangents des surfaces P_1, P_2. Dans le travail présent l'auteur a résolu le problème suivant: Etant données les surfaces P_1, P_2, on doit déterminer les autres couples des surfaces, qui ont les mêmes invariants (1).

A l'aide des méthodes de la géometrie différentielle classique, des propriétés de l'affinité entre les vecteurs tangents et par l'intégration des équations différentielles partielles du type fondamental, l'auteur prouve le théorème fondamental, c'est-à-dire la condition nécessaire et suffisante pour l'existence des autres couple des surfaces, qui possèdent les invariants (1) avec les surfaces P_1, P_2 communs.

Le cas des surfaces réglées est aussi mentionné.

V. Vilhelm (Praha).

Prvanovitch, Mileva. Hyper-Darboux lines on a surface in three dimensional Euclidean space. Bull. Calcutta Math. Soc. **47** (1955), 55–60.

Par analogie avec les courbes d'union d'une congruence quelconque de courbes G relativement à une surface S de l'espace euclidien à trois dimensions, l'auteur définit les hyper-lignes de Darboux de S, qui sont celles dont le rayon de la sphère osculatrice au point courant est porté par la tangente à la courbe de la congruence G issue de ce point. Il forme l'équation différentielle des hyper-lignes de Darboux d'une surface quelconque, et signale les cas où ces courbes se réduisent aux lignes de Darboux ordinaires.

P. Vincensini (Marseille).

Kaul, R. N. Formulae corresponding to Frenet's formulae. Acad. Roy. Belg. Bull. Cl. Sci. (5) **41** (1955), 1292–1304.

Dans cet article l'auteur étudie, dans un espace riemannien à n dimensions, les formules correspondant aux formules ordinaires de Frenet, déduites de la considération du vecteur unitaire normal d'une hypersurface de l'espace envisagé. Il forme un ennuple orthogonal de vecteurs, dont l'un est le vecteur unitaire normal à l'hypersurface, les autres étant tangents à celle-ci, et il développe les propriétés des dérivées intrinsèques de ces différents vecteurs le long d'une courbe quelconque de l'hypersurface considérée. *P. Vincensini* (Marseille).

Pan, T. K. Centers of curvatures of a vector field. Proc. Amer. Math. Soc. **7** (1956), 292–298.

The author considers a unit vector field on a surface in three-dimensional euclidean space. The derived vector with respect to a curve, and the components of this derived vector in the tangential and normal planes defined at any point of the surface, lean to the notions of absolute curvature, associate curvature and normal curvature. In this paper various theorems are proved concerning the corresponding centers of curvature.

E. T. Davies (Southampton).

Manara, Carlo Felice. Osservazioni sulla geometria delle equazioni differenziali nel piano complesso. Accad. Sci. Modena. Atti Mem. (5) **13** (1955), 252–257.

The independent and dependent variables being real and complex, respectively, the differential equation $d^2z/dt^2=w(t)dz/dt$ can be regarded as defining a plane curve to within a similitude $z'=az+b$, where a and b are arbitrary complex constants. In this note a few of the elementary parts of the theory of plane curves are developed from this point of view. *L. A. MacColl.*

Kautny, Walter. Zur Geometrie des harmonischen Umschwungs. Monatsh. Math. **60** (1956), 66–82.

Harmonischer Umschwung ist der Bewegungsvorgang, der sich aus einer Drehung um eine feste Achse und einer harmonischen Schwingung längs dieser Achse zusammensetzt. In Zylinderkoordinaten: $r=r_0$, $\varphi=t$, $z=h\sin nt$. Für $n=1$ hat man ein Sonderfall der Mannheim-Darbouxsche Bewegung. Die Bahnkurven für $n\neq 1$ werden untersucht, auch mittels darstellender Geometrie. Für n rational sind sie algebraisch. Die von einer Ebene erzeugte Hülltorse ihrer Gratlinie ist eine Böschungslinie.

O. Bottema (Delft).

Demaria, Davide Carlo. Sui sistemi di curve iperspaziali che godono della proprietà proiettiva o prospettiva in prima approssimazione. Mem. Accad. Sci. Torino. Cl. Fis. Mat. Nat. (3) **1** (1955), 69–82.

A system of differential equations of the form

$$
y'''=\varphi(x, y, z, y', z', y'', z'')
$$
$$
z'''=\psi(x, y, z, y', z', y'', z'')
$$

defines a six-parameter family of curves in 3-space. In this paper it is shown that a necessary and sufficient condition for the family of curves to possess a certain property, called the projective property in the first approximation, is that the system of differential equations be of the special form

$$
y'''=Ay''^2+By''z''+Cy''+Dz''+E
$$
$$
z'''=Bz''^2+Ay''z''+C'y''+D'z''+E',
$$

where A, B, C, D, C', D', E' are functions of x, y, z, y', z'. Similarly, it is shown that for the family of curves to possess a certain "prospective property in the first approximation" it is necessary and sufficient that the system be of the form

$$
y'''/y''=z'''/z''=Ay''+Bz''+C.
$$

These results are generalized to the case of a $3k$-parameter family of curves in $(k+1)$-space. *L. A. MacColl.*

Özkan, Asim. Sur les tissus quadruples hexagonaux se trouvant sur une surface, formés de familles de courbes parallèles. Rev. Fac. Sci. Univ. Istanbul. Sér. A. 20 (1955), 105–111 (1956). (Turkish summary)

Let w^k denote differentiable k-webs of curves in the plane or on a surface S. If the ith family of curves is omitted from w^4, a 3-web w_i^3 is obtained, w^4 is "hexagonal" if all its w_i^3 are hexagonal. The curves of each family of a "parallel" w^k on S are parallel. Contents: (i) The w^4 are determined for which exactly three w_i^3 are hexagonal and Mayrhofer's classification of the hexagonal w^4 is obtained [Math. Z. 28 (1928), 728–752]. (ii) The first fundamental forms of those S are computed which possess parallel hexagonal w^3. (iii) The author proves: If three of the w_i^3 of a parallel w^4 on S are hexagonal, w^4 is hexagonal. (iv) The S are determined which have parallel hexagonal w^4. Those S belong to three types corresponding to Mayrhofer's three classes of plane hexagonal w^4.

P. Scherk (Kingston, Ont.).

Godeaux, Lucien. Sur la théorie des congruences W. III. Acad. Roy. Belg. Bull. Cl. Sci. (5) 42 (1956), 240–244.

[Pour les parties I–II voir même Bull. (5) 40 (1954), 1028–1037; 41 (1955), 343–345; MR 16, 855, 856.] Dans cette note l'auteur substitue, pour l'étude des congruences W, au tétraèdre mobile de E. Cartan attaché à l'un des foyers d'une congruence rectiligne, un tétraèdre mobile intrinsèquement lié au rayon générateur de la congruence dans la définition duquel les deux foyers jouent le même rôle. Il en résulte une symétrie dont peuvent bénéficier les expositions de bien des résultats de la théorie des congruences W. x et $\bar{x}$ étant les deux foyers d'une droite j d'une congruence W, les quadriques de Lie Φ et $\bar{\Phi}$ attachées à x et $\bar{x}$ ont en commun un quadrilatère gauche. Soient r_1 et r_2 les droites formant un tétraèdre avec les cotés de ce quadrilatère; le produit des polarités relatives à Φ et $\bar{\Phi}$ est une homographie biaxiale d'axes r_1 et r_2. Si s_1 et s_2 sont les droites s'appuyant sur r_1 et r_2 passant respectivement par x et $\bar{x}$, et si $\bar{m}$ et $\bar{n}$ sont les points des droites précédentes respectivement conjugués harmoniques de x et $\bar{x}$ par rapport aux points où ces droites s'appuient sur r_1 et r_2, le tétraèdre en question est $x\bar{x}\bar{m}\bar{n}$. L'auteur établit les équations symétriques définissant les quadriques associées à la congruence W envisagée, de même que les équations de l'homographie biaxiale d'axes r_1 et r_2 produit des polarités par rapport aux quadriques Φ et $\bar{\Phi}$.

P. Vincensini (Marseille).

Franckx, Ed. Surfaces réglées gauches applicables. Bull. Soc. Roy. Sci. Liège 25 (1956), 116–118.

Wintner, Aurel. On Weyl's imbedding problem. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 157–160.

The purpose of the paper is an improvement of a result by Nirenberg [Comm. Pure Appl. Math. 6 (1953), 337–394; MR 15, 347]. However, the result is already contained in Nirenberg's paper; the misunderstanding derives, according to a communication from the author, from a difference in notation.

H. Busemann.

Nitsche, Joachim. Beiträge zur Verbiegung zweifach zusammenhängender Flächenstücke. Math. Z. 62 (1955), 388–399.

This paper is concerned with the bending of the zone on a sphere cut out by two parallel planes. The chief problem considered is to prescribe as constants the curvatures ϱ_1 and ϱ_2 of the two boundary curves of the bent surface. When the planes are equidistant from the center of the sphere, no solution exists, if $\varrho_1 \neq \varrho_2$, while there is a one parametric family of solutions, if $\varrho_1 = \varrho_2$. When the planes are at different distances from the center of the sphere, there may be just two solutions, just one solution, or none, according to the sign of a simple expression. The results thus contrast with those previously obtained by the author [Arch. Math. 4 (1953), 331–336; 6 (1955), 13–17; MR 15, 446; 16, 955] for the bending of a simply connected surface. Any bent surface in the simply connected case is completely determined by the curvature of its boundary, and, moreover, a surface exists corresponding to any prescribed curvature, provided three compatibility conditions of an integral character hold.

A. Douglis (New York, N.Y.).

Kobayashi, Shôshichi. On connections of Cartan. Canad. J. Math. 8 (1956), 145–156.

Let P be a differentiable principal fiber bundle with group G over a base manifold M, and let B be an associated bundle with fiber F, which admits a cross section $\sigma: M \to B$. The group G is assumed to be transitive on F; G' is the stationary group of a point of F (hence $F = G/G'$), and B' is the bundle B with group G'; and P' is the principal bundle of B', which is considered as a subbundle of P.

The bundle B is said to be "soudé" to M if there is given a rule which identifies the tangent space $T_{\sigma(x)}(F_x)$ with $T_x(M)$. (Such a rule does not exist in general, of course.)

The homogeneous space $F = G/G'$ is weakly reductive if the Lie algebra $\mathfrak{g}$ of G admits a direct sum decomposition (as a vector space): $\mathfrak{g} = \mathfrak{g}' + \mathfrak{f}$, where $\mathfrak{g}'$ is the Lie algebra of G', and where $[\mathfrak{g}', \mathfrak{f}] \subset \mathfrak{f}$ is satisfied. (Examples: G' is (1) compact, or (2) semi-simple and connected, or (3) discrete.) In this case, $\mathfrak{f}$ can be identified with the tangent space to F at the point $\{e\}$.

Let there be given a connection in B' (or P'), then the author constructs a "Cartan connection" in B (and P) which generalizes the Cartan connection ("rolling" process) in the tangent bundle, associated with an affine connection. The assumptions are that B is soudé to M, and that F is weakly reductive.

A connection in B (or B') is given by prescribing a family of right-invariant horizontal planes in P (or P'); or, equivalently, by defining over P (or P') a $\mathfrak{g}$-valued (or $\mathfrak{g}'$-valued) differential form ω(or ω') satisfying some conditions. Let u be a tangent vector to P, at a point aG_x of P', then u can be decomposed uniquely into $u' + u''$, where u' is tangent to P', and $u'' \in a \cdot \mathfrak{f}$. (Remember that a is a mapping $G \to G_x$ which can be extended to tangent vectors to these spaces.) The form ω' is extended to u by putting $\omega(u) = \omega(u')$. — Define the $\mathfrak{f}$-valued differential form θ on P' for tangent vectors to P by assigning to u the tangent vector to M onto which it projects under the map $G_x \to x$; because B is soudé to M, this is an element of $T_{\sigma(x)}(F_x)$, and because F is weakly reductive this gives an element of $\mathfrak{f}$, which by definition is $\theta(u)$. The form ω of the Cartan connection is defined as $\omega = \omega' + \theta$; first for vectors tangent to P at points of P', and then extended uniquely to all points of P by the invariance conditions for forms of connections.

Let Ω' be the curvature form of ω', Ω that of ω, and let $\Omega_{\mathfrak{g}'}$ and $\Omega_{\mathfrak{f}}$ be the parts of Ω in the decomposition. Then Ω' and $\Omega_{\mathfrak{g}'}$ coincide if $[\mathfrak{f}, \mathfrak{f}]_{\mathfrak{g}'} = 0$. The torsion form $\Theta =$

$\Omega_{\mathfrak{f}}-\frac{1}{2}[\theta,\theta]_{\mathfrak{f}}$ equals $\Omega_{\mathfrak{f}}$ if $[\mathfrak{f},\mathfrak{f}]_{\mathfrak{f}}=0$. These latter conditions are satisfied for the Cartan connections of tangent bundles.

The paper is concluded with a section on invariant Cartan connections, for which case curvature, torsion and the Lie algebra of the holonomy group are computed.

A. Nijenhuis (Seattle, Wash.).

Guggenheimer, H. Plurigenera of complex manifolds. Bull. Res. Council Israel Sect. A. **5** (1955), 20–22.

Les plurigenres sont définis sur des variétés à structure complexe, à partie des formes différentielles de type $(m_1, n_1; m_2, n_2; \cdots; m_s, n_s)$ les différentielles dz_i, $d\bar{z}_j$ étant répartées en s systèmes: deux différentielles d'un même système sont indépendantes; elles subissent la multiplication extérieure; on ne fait aucune hypothèse sur deux différentielles de systèmes différents. On considère le groupe $H(m_1, n_1; \cdots; m_s, n_s)$ quotient du groupe des formes $\varphi_{m_1,n_1;\cdots;m_s,n_s}$ pour lesquelles $d'\varphi=d''\varphi=0$ par le groupe $B(m_1, n_1; \cdots; m_s, n_s)$ des formes qui sont des $d'd''\psi$, ou des $d'\psi$ (si $n_i=0$ pour un $m_i\neq0$) ou des $d''\psi$ (dans le cas où il existe $m_i=0$ avec $n_i\neq0$). Le rang de ce groupe pour $m_1=m_2=\cdots=m_s=k$ et $n_1=n_2=\cdots=n_s=0$ est appelé le plurigenre d'ordre s de la variété M de dimension complexe k; il coincide avec celui défini par Castelnuovo et Enriques dans le cas où M est algébrique; la notion est invariante par les modifications analytiques.

P. Lelong (Paris).

Prvanovitch, Mileva. Lignes de Darboux dans l'espace riemannien. Bull. Sci. Math. (2) **78** (1954), 9–14.

Dans un travail antérieur [Rev. Fac. Sci. Univ. Istanbul (A) **19** (1954), 13–18; MR **16**, 400], l'auteur a défini les lignes de Darboux de l'espace riemannien V_n plongé dans l'espace euclidien E_m comme la généralisation naturelle des lignes de Darboux de l'espace ordinaire, à savoir les courbes de V_n tangentes en chacun de leurs points à l'hypersphère de E_m osculatrice en ce point. Il envisage ici les lignes de Darboux d'un sous-espace V_n d'un espace riemannien V_m, forme leurs équations différentielles, et indique quelques conséquences relatives aux lignes géodésiques de V_n, à certains transports parallèles dans V_n, au cas où V_n est totalement géodésique dans V_m, et à celui où V_m se réduit à l'espace euclidien ordinaire.

P. Vincensini (Marseille).

Fabricius-Bjerre, Fr. Eine Darstellung von J. Hjelmslevs projektiver Infinitesimalgeometrie. Acta Math. **95** (1956), 111–154.

The definitions are much too involved for reproduction in a review, so that only a description of the topics treated can be given. The purpose is a study of the curvatures of curves in P^2 and P^3 and surfaces in P^3, thus there is very little relation to projective differential geometry in the usual sense, because the latter deals with invariants depending on derivatives of higher order than 2.

First, elements of different orders are defined for curves in P^3, also curvatures (depending on auxiliary points). The ∞^2 conics having second order contact with a plane curve at a given point replace the osculating circle. These examples are dualized to line curves in a plane and to one-parameter families of lines or planes through a point. There follows an investigation of envelopes from the projective point of view which is attractive through the geometric reasoning and its rigor. Then comes a projective analogue of Meusnier's Theorem. Let two spatial curves C_1, C_2 intersect at p with different tangents, and let there be a correspondence between C_1 and C_2 under which p corresponds to itself. The behavior at p of the ruled surface obtained by connecting corresponding points of C_1 and C_2 is studied in detail. The dual results on the intersection of two cones are also interesting.

Ruled surfaces are discussed in much greater detail than usual. The paper ends with an investigation of general surfaces centering about the behavior of the curvatures of plane sections.

H. Busemann.

Ščerbakov, R. N. Some questions of the affine theory of rectilinear congruences. Mat. Sb. N.S. **37(79)** (1955), 527–556. (Russian)

Dans le mémoire on emploi la méthode du repère mobile et les formes extérieurs à l'étude des problèmes des congruences des droites dans la géométrie différentielle équiaffine. Dans le premier chapitre l'auteur définit le repère $\{A, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ invariant (équiaffin) pour la congruence non parabolique. On choisit le point A et les vecteurs $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ afin que les points focaux soient $F_{12}=A\pm\mathbf{e}_3$ et les développables soient les surfaces réglées coordonnées. On peut choisir le plan $\{\mathbf{e}_1, \mathbf{e}_2\}$ dans le faisceau des plans déterminés par le plan tangent de la surface centrale (S-repère) et le plan défini par les directions conjuguées aux droites de la congruence qui sont situées sur les surfaces focaux (F-repère). L'axe du faisceau est la droite canonique analogue à la droite canonique de la théorie projective des surfaces. En partant du paramètre de ce faisceau on peut définir le t-repère.

Les invariants fondamentaux de la congruence générale sont déterminés. On étudie aussi les congruences spéciales définies par les équations $I_f=0$, où I_f est l'invariant fondamental.

Ensuite on établie la correspondance entre la congruence et la congruence de ses droites canoniques. Si les développables de deux congruences sont en correspondance, la congruence mentionée a les propriétés suivantes: 1) une surface focale est la courbe. 2) Les développables coupent la surface centrale dans le réseau conjugué. 3) aux développables correspond le réseau conjugué sur la surface indicatrice. La congruence ayant deux des trois propriétés mentionées possède aussi la troisième et dépend de 5 fonctions à un paramètre. L'auteur étudie aussi les congruences spéciales qui dépendent de 5 fonctions à une variable et pour lesquelles la normale affine d'une des surfaces focales dans le point arbitraire est parallèle à la direction conjugée à la tangente de l'arrêt de rebroussement des développables sur la seconde surface focale. Le second chapitre est consacré à l'étude des surfaces réglées de la congruence. On établit les invariants fondamentaux et leurs interprétations géometriques. Ensuite on étudie la transformation E de M. Egorov, c'est-à-dire la congruence C est considérée comme l'ensemble (∞^1) des surfaces réglées dont chacune se transforme à l'aide d'une affinité. Ainsi on aura une nouvelle congruence C_1. Si les surfaces centrales des C et C_1 sont en correspondance, on a la transformation E de la 1ère espèce; si le réseau conjugué sur les surfaces réglées se transforme dans le réseau conjugué, on a la transformation E de la 2ème espèce. Dans la congruence donnée il y a ∞^{12} surfaces réglées à l'aide desquelles on peut définir la transformation E de la 1ère ou 2ème espèce. Les divers cas des transformations E sont montrés. Enfin on démontre l'interprétation géométrique des invariants fondamentaux de la congruence en utilisant les invariants des surfaces réglées dépendantes d'un paramètre et situées dans la congruence.

F. Vyčichlo (Prague).

Širokov, A. P. On a property of covariantly constant affinors. Dokl. Akad. Nauk SSSR (N.S.) 102 (1955), 461–464. (Russian)

On étudie l'affineur f_β^α avec la matrice $f=\|f_\beta^\alpha\|$ pour lequel dans la connexion $\Gamma^\gamma_{\alpha\beta}$, sans torsion, on a $\nabla_\alpha f_\gamma^\beta=0$. On peut démontrer les théorèmes: 1) Les nombres caractéristiques λ de l'affineur covariantement constant sont les constantes. 2) Les surfaces construites sur les vecteurs $\mathbf{v}$ satisfaisants à l'équation $(f-\lambda E)^k\mathbf{v}=0$ sont les figures des surfaces absolument parallèles. 3) On peut construire le système holonome des coordonnées dans lequel les composantes de l'affineur covariantement constant sont les constantes. (La matrice de l'affineur peut être amenée au type canonique de Jordan.)

La démonstration constructive du 3ème théorème est montrée sur l'exemple de l'affineur nilpotent dont la caractéristique contient seulement la chiffre 2 et qui est située dans l'espace à $2n$ dimensions et aussi sur l'exemple de l'affineur avec le déterminant $|f^\alpha_\beta|=0$. *F. Vyčichlo.*

Kimpara, Makoto. Une formule de Stokes dans un espace projectif. Tensor (N.S.) 5 (1955), 123–126.

Let S be a hypersurface of real projective $(n+1)$-space. The author defines an invariant linear differential form σ, and invariant quadratic forms Ω and Π such that $d\sigma=\Omega+\Pi$. He then proves the theorem: Let D be a simply-connected two dimensional domain in S with boundary D. Suppose that on D a certain determinant, Δ, does not vanish and that the hyperquadric of Wilczynski does not coincide with that of Lie. Then

$$\int_C \sigma=\int_D\int \Omega+\Pi.$$

C. B. Allendoerfer (Seattle, Wash.).

Mineo, Corradino. Geodesia intrinseca e proprietà generali delle rappresentazioni cartografiche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 18 (1955), 569–576.

Riemannian Geometry, Connections

Petkantschin, B. Parabolische Regelscharen in der zweiachsigen Geometrie. C. R. Acad. Bulgare Sci. 8 (1955), no. 1, 1–4. (Russian summary)

Dans ce travail l'auteur complète les considérations publiées dans la mémoire analysée ci-dessus et détermine les invariants du système des droites $g(t)$ à un paramètre en supposant que les droites fondamentales j, k de la géometrie biaxiale et les droites $g(t)$, $g^*(t+h)$ pour $h\to 0$ possèdent une seule axe. Il obtient aussi le paramètre invariant s du système et deux invariants L, M fondamentaux, lesquels déterminent avec les conditions initiales le système $g(s)$ univoquement jusqu'aux homographies qui transforment la droite j dans j et la droite k dans k. *F. Vyčichlo* (Prague).

Petkantschin, Bojan. Hyperbolische Regelscharen in der zweiachsigen Geometrie. Ann. Univ. Sofia Fac. Sci. Phys. Math. Livre 1. 48 (1953/54), 33–67 (1954). (Bulgarian. German summary)

Soient j, k deux droites qui ne se coupent pas dans l'espace réel projectif P_3. On étudie les objects et les propriétés invariantes par rapport aux homographies dans P_3, qui transforment la droite j dans la même droite et aussi la droite k ne changent pas. (La géométrie biaxiale). Specialement l'auteur considère le système à un paramètre des droites $g(t)$, $t\in(t_1, t_2)$.

Il suppose que les droites j, k, $g(t)$, $g^*(t+h)$ possèdent deux droites réelles, diverses (les axes) qui les coupent aussi dans la position limite pour laquelle $h\to 0$. Les points de la droite $g(t)$ comun aux axes (les points doubles de la droite) sont invariants. L'auteur détermine le paramétre s invariant et les trois invariants fondamentaux du système $g(t)$ et montre leurs interprétations géométriques. Les invariants mentionés et les conditions initiales déterminent le système $g(s)$ univoquement jusqu'à la homographie considérée. La condition nécessaire et suffisante pour que le système $g(t)$ soit dévelopable est aussi trouvée. Pendant l'étude l'auteur emploie le système coordonné dont deux points fondamentaux sont situés sur la droite j et deux points sur la droite k. Les coordonnés (x_1, x_2, y_1, y_2) du point on écrit (x, y), où x signifie le couple ordonné (x_1, x_2), y le couple (y_1, y_2). Dans le calcul on emploie le produit $xy=x_1y_2-x_2y_1$ (les coordonnées plückeriennes). *F. Vyčichlo* (Prague).

Fukami, Tetsuzo; and Ishihara, Shigeru. Almost Hermitian structure on S^6. Tôhoku Math. J. (2) 7 (1955), 151–156.

It is well-known that an almost Hermitian structure can be defined on the 6-dimensional sphere S^6 by making use of the algebra C of Cayley numbers [Frölicher, Math. Ann. 129 (1955), 50–95; MR 16, 857]. The group G of all automorphisms of the algebra C acts transitively on S^6 as a group of isometries. The authors show that the above almost Hermitian structure on S^6 is invariant under the group G. They show, further, that there exists one and only one affine connection on S^6 invariant under G and such that the covariant derivative of the almost Hermitian metric tensor and the almost complex structure tensor are constant. A property of this connection is that its curvature and torsion tensor fields are covariantly constant. *L. Auslander* (Princeton, N.J.).

Ishihara, Shigeru; and Obata, Morio. Affine transformations in a Riemannian manifold. Tôhoku Math. J. (2) 7 (1955), 146–150.

The authors in this paper study the relation between the group $A(M)$ of all affine structure preserving transformations of the affine structure induced by a Riemann metric and the group $I(M)$ of all isometries of the metric. They prove that if M is a complete connected irreducible Riemann manifold then $I(M)=A(M)$. Let $A_0(M)$ and $I_0(M)$ denote the identity components of $A(M)$ and $I(M)$ respectively. Then if M is a complete connected Riemann manifold which is nowhere locally flat, then $A_0(M)=I_0(M)$. At the end of the paper are two interesting examples to show that the above hypotheses cannot be weakened. *L. Auslander* (Princeton, N.J.).

Kostant, Bertram. On invariant skew-tensors. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 148–151.

The symbol $\{\sigma, \mathfrak{k}, V\}$ denotes a representation of a Lie algebra $\mathfrak{k}$ (of a compact group K) on an n-dimensional vector space V with positive definite inner product. The linear mappings $\sigma(X)$, $X\in\mathfrak{k}$ are all assumed to be skew-symmetric with respect to the inner product on V. By W_p we denote the set of all those skew-tensors s over V of degree p which satisfy $\sigma(X)\cdot s=0$ for all $X\in\mathfrak{k}$, and $d_p=\dim W_p$. The Poincaré polynomial of $\{\sigma, \mathfrak{k}, V\}$ is defined as $P(t)=\sum_p d_pt^p$.

A representation $\{\sigma, \mathfrak{k}, V\}$ is symmetric if K is a subgroup of a Lie group G, and if $M=G/K$ is a compact homogeneous Riemannian manifold, G being the identity component of the full group of motions of M. Then $\{\sigma, \mathfrak{k}, V\}$ must be the obvious representation of $\mathfrak{k}$ on the tangent space to M at the point represented by K. The symmetric representations have been classified. If a representation is also irreducible, it is called an S-representation, and if also K is simple (including the cases of dim $K=0, 1$), it is called a T-representation. There are two types of T-representations: the T_1-representations are adjoint representations of the simple non-abelian compact Lie algebras, and the T_2-representations are non-adjoint representations, but can be enumerated in terms of their homogeneous spaces M.

Theorem 1 states that if $\{\sigma, \mathfrak{k}, V\}$ is faithful, irreducible, then $d_4=0$ is equivalent with $\{\sigma, \mathfrak{k}, V\}$ being a T-representation, and Theorem 1′ says that if $\{\sigma, \mathfrak{k}, V\}$ is faithful, then $d_4=0$ if and only if $\{\sigma, \mathfrak{k}, V\}$ is a composite of T-representations $\{\sigma_1, \mathfrak{k}_i, V\}$ such that the coefficient of t^4 in $\prod_i P_{\sigma_i}(t)$ is zero. — The second statement is proved using that $d_3=0$ for T_2-representations alone; $d_2=0$ for non-circular representations, and $d_1=0$ for non-trivial representations. — Theorem 2 states that if $\{\sigma, \mathfrak{k}, V\}$ is faithful and $P_\sigma(t)=1+t^n$, then either $\{\sigma, \mathfrak{k}, V\}$ corresponds to $SO(n)/SO(n-1)$ or to $SU(3)/SO(3)$.

These results are applied to holonomy groups. Let M be a simply connected, irreducible manifold and let there be no covariant constant 4-forms. Then the holonomy representation is a T-representation. If M is not Riemannian symmetric, then $\{\sigma, \mathfrak{k}, V\}$ corresponds to $SO(n)/SO(n-1)$ or to $F_4/\mathrm{Spin}(9)$. The latter case ($n=16$) can occur only if M admits a covariant constant 8-form. Another result, generalizing theorems of Iwamoto and of Singer, is that if M is compact, simply connected, and is topologically a rational homology sphere, then the holonomy representation corresponds to $SO(n)/SO(n-1)$ or to $SU(3)/SO(3)$. The latter case occurs only if M is locally isometric to $SU(3)/SO(3)$. *A. Nijenhuis.*

Leichtweiss, Kurt. Das Problem von Cauchy in der mehrdimensionalen Differentialgeometrie. I. Zur isometrischen Einbettung und Verbiegung von Riemannschen Räumen. Math. Ann. **130** (1956), 442–474.

Results on local isometric imbedding of Riemannian spaces similar to those previously given by E. Cartan [Ann. Soc. Pol. Math. **6** (1928), 1–7], and by C. Burstin [Mat. Sb. **38** (1931), no. 3–4, 74–85] are obtained by a new method which yields further information in the case of analytic imbedding.

Let R_m be an m-dimensional manifold with local coordinates (u^c) of origin at P_0, and let $g_{ab}(u^c)$ be a positive definite metric which is analytic at P_0 $(a, b, c=1, 2, \cdots, m)$. Let R_n be an n-dimensional manifold with local coordinates (x^k) with origin Q_0, and let $g_{ij}(x^k)$ be a positive definite metric analytic at Q_0 $(i, j, k=1, 2, \cdots, n)$. Then if $n\geq\frac{1}{2}m(m+1)$, it is proved that R_m can be locally, analytically and isometrically imbedded in R_n so that P_0 falls on Q_0.

The author then gives theorems on the relation between two m-dimensional manifolds each of which is isometrically imbedded in R_n, showing that conslusions in the cases $n>\frac{1}{2}m(m+1)$ and $n=\frac{1}{2}m(m+1)$ are essentially different.

The theorem of Cauchy-Kowaleski is used to prove the existence and uniqueness of imbeddings. Classical tensor calculus is used throughout and the Cartan calculus of exterior forms is enteintionally avoided. The imbdding problems considered in this paper are essentially local in character. *T. J. Willmore* (Liverpool).

Debever, R. Sur un théorème de B. Segre. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 26–27.

A new proof is given of the following theorem of B. Segre [same Rend. (8) **7** (1949), 12–15; MR **11**, 541]: A Riemannian space V_n represented conformally on euclidean space S_n so that geodesics of V_n correspond to curves with constant curvature in S_n, is itself of constant curvature. *T. J. Willmore* (Liverpool).

Sulanke, Rolf. Die eindeutige Bestimmtheit des von Hanno Rund eingeführten Zusammenhangs in Finsler-Räumen. Wiss. Z. Humboldt-Univ. Berlin. Math.-Nat. Reihe **4** (1954/55), 229–233. (Russian, English and French summaries)

This paper is closely related to the work of Rund [Math. Ann. **125** (1952), 1–18; **127** (1954), 82–102, MR **14**, 499; **15**, 898] and of Bompiani [Arch. Math. **3** (1952), 183–186; MR **14**, 409]. The first part of the paper is concerned with an affine connection in a space where the coefficients of connection depend on a field of directions. Motions depending on this field of directions are referred to as "relative". On the other hand the word "absolute" is applied to sections when they refer to a particular field of directions which is "stationary" at a line element.

In the second part of the paper the author shows how the absolute connection coefficients used by Rund in Finsler spaces can be obtained by using three postulates: (i) the symmetry of the connection coefficient, (ii) the coincidence of the extremals of the variations problem with the absolute geodesics, (iii) the vanishing of the absolute covariant derivative of the fundamental metric tensor. The connection coefficients $P_{jk}{}^i$ finally obtained by the author are in fact identical with the $\mathring{\Gamma}_{jk}{}^i$ of E. Cartan [Les espaces de Finsler, Hermann, Paris, 1934, p. 16]. *E. T. Davies* (Southampton).

Complex Manifolds

Zisman, Michel. Algèbre caractéristique projective des variétés presque-complexes. C. R. Acad. Sci. Paris **242** (1956), 2436–2438.

Let V_{2n} be a differentiable almost-complex manifold, $V_{2n}\times R^2$ be given a canonical almost-complex structure, and $E'\to V_{2n}$ be the principal bundle with group $GL(n+1, C)$ defined by restricting the principal bundle associated to the tangent bundle of $V_{2n}\times R^2$ to the submanifold $V_{2n}\times\{0\}$, identified with V_{2n}. If $\mathcal{E}'$ is the bundle subordinate to E' with group $U(n+1)$, and if $\mathcal{C}$ is the center of $U(n+1)$, then $\mathcal{E}=\mathcal{E}'/\mathcal{C}\to V_{2n}$ is a principal bundle with group $PU(n)$, the projective unitary group. The characteristic algebra of the bundle $\mathcal{E}\to V_{2n}$ is determined by the Chern classes of the almost-complex manifold V_{2n}, the precise expression of this determination being the main result of the paper; this algebra actually coincides with the characteristic algebra of V_{2n} if and only if the second Chern class of V_{2n} is zero. *R. C. Gunning.*

Martin, W. T.; Chern, S. S.; and Zariski, Oscar. Scientific report on the Second Summer Institute, several complex variables. Bull. Amer. Math. Soc. **62** (1956), 79–141.

Sous le titre „variables complexes", on rend compte

ici d'un très vaste ensemble de recherches où intervient la structure complexe. La partie I est consacrée à l',,Analyse''; II aux variété et espaces fibrés; la partie III, qui est la plus étendue, concerne la théorie des faisceaux et les applications de la cohomologie en géométrie algébrique. On se contentera ici de donner des indications succinctes sur ces trois articles; la partie III donne un exposé très informé des nouvelles méthodes utilisées en géométrie algébrique; on regrette seulement que le compte-rendu d'un congrès de Juillet 1954 ne paraisse imprimé qu'en Mars 1956.

La partie I est restreinte; elle mentionne des résultats concernant: les fonctions de carré sommable, le noyau de S. Bergmann, les domaines minimaux, les représentations des fonctions dans un polyèdre analytique ou dans un domaine à frontière remarquable du type étudié par S. Bergmann. Concernant les fonctions plurisousharmoniques, on donne le résultat suivant (Bremermann): soit $u(M)$ une donnée continue sur l'arête A d'un polyèdre analytique P: l'enveloppe supérieure des fonctions plurisousharmoniques dans P majorées par $u(m)$ sur A est une fonction plurisousharmonique égale à $u(M)$ sur A. [Une erreur du texte doit être ici rectifiée: pour que V $(-\infty\leqq V<+\infty)$ soit plurisousharmonique, il faut et il suffit a) que V soit localement sommable, b) que la distribution $\delta[V, \boldsymbol{\lambda}]=\sum V^{ij}\lambda_i\bar{\lambda}_j$, $V^{ij}=\partial^2V/\partial z_i\partial\bar{z}_j$; soit une mesure positive quel que soit le vecteur $\boldsymbol{\lambda}$ complexe, c) que $V(P)=V_m(P)$, $V_m(P)$ étant le maximum en mesure de V au point P; a), b), c) entrainent la propriété c_1): V est semi-continue supérieurement, mais a), b), c_1) n'entrainens pas que V soit plurisousharmonique. Des résultats sur let fonctions automorphes (Baily) figurent dans cette partie, qui résume encore les études (Fueter, Haefeli) sur les fonctions dans une algèbre, et se termine par l'étude d'opérateurs, introduits par S. Bergmann, qui font passer d'une fonction holomorphe de C^2 à une fonction harmonique de R^3. La rédaction de cette partie I est fort peu synthétique; des résultats essentiels, obtenus par des méthodes analytiques (et cités dans II), n'y ont pas trouvé place; c'est curieusement, comme un ,,lemme'' qu'est cité le théorème de K. Oka [Tôhoku Math. J. **49** (1942), 15–52; Jap. J. Math. **23** (1953), 97–155; MR **7**, 290; **17**, 82]: un domaine pseudo-convexe est un domaine d'holomorphie.

La partie II résume les travaux récents faits sur les variétés à structure presque complexe ou complexe: conditions d'existence de telles structures, classes de Chern, cas de S^{2n}. Problèmes concernant les espaces fibrés, affines ou projectifs, ou à fibre vectorielle; des travaux non publiés à l'époque sont résumés ici (Grothendieck, Nakano et Serre); travaux de Atiyah sur la classification des fibrés projectifs dont la base est une courbe algébrique. Formulation dans le langage de la cohomologie est des faisceaux des problèmes de Cousin; étude des groupes $H^p(M, \Omega^p)$, Ω^p étant le faisceau des germes des formes $(p, 0)$ à coefficients holomorphes sur la variété M. Sont résumés les travaux de Serre, Kodaira, Spencer, et à cette occasion sont soulignées les particularitiés propres aux variétés kähleriennes, algébriques ou de Stein (notamment les théorèmes A et B de Cartan-Serre, obtenus à partie de résultats de K. Oka). Formulation du théorème de Riemann-Roch à la suite des travaux de Hirzebruch; on indique le lieu du résultat concernant l'index avec les travaux de Thom. Cette partie mentionne aussi les résultats récents sur les espaces homogènes et esquisse des généralisations où la structure complexe est remplacée par un pseudo-groupe au sens de Lie et E. Cartan.

La partie III expose la théorie des faisceaux et les applications de la cohomologie en géométrie algébrique abstraite: notion générale de faisceau; faisceau d'opérateurs; exemples; topologie de Zariski. Notion de faisceau cohérent; groupe de cohomologie et suites exactes; homomorphisme algébrique d'un faisceau cohérent de base une variété affine, sur un autre faisceau. Faisceaux semi-finis (au sens de Serre) sur une telle variété. Faisceaux sur une variété projective. On indique l'adaptation des méthodes classiques aux problèmes formulés en termes abstraits, et aussi, plus brièvement, le rôle des foncteurs, selon les méthodes de Serre. Travaux sur le genre arithmétique. Discussion, à propos du théorème de Riemann-Roch et de ses généralisations des apports de différentes méthodes (Zariski, Serre, Kodaira). Résultats de Serre sur les faisceaux algébriques cohérents [Ann. of Math. (2) **61** (1955), 197–278; MR **16**, 953]. *P. Lelong* (Paris).

Algebraic Geometry

Gönenç, Süeda. Problemi in connessione colle curve di Bertrand. Rev. Fac. Sci. Univ. Istanbul. Sér. A. **20** (1955), 141–147 (1956). (Turkish summary)

Author considers in hyperbolic space two ruled surfaces or two skew curves of which the moving trihedra in corresponding points have fixed distances and fixed angles. In the case of curves there is a linear relation between curvature and torsion. The method is analytic, making use of quaternions. *O. Bottema* (Delft).

Fischer, Irwin. The moduli of hyperelliptic curves. Trans. Amer. Math. Soc. **82** (1956), 64–84.

The problem of the moduli of algebraic curves is formulated as follows. Let the totality of irreducible curves of fixed genus g defined over a given ground field k be partitioned into classes by the relation of birational equivalence. Is it possible to construct a 1:1 mapping π of these classes into the set of algebraic points of a variety N (the moduli-variety) defined over k, in such a way that the complement of the set of images is a proper algebraic subvariety of N? The inadequacy of van der Waerden's treatment of the general problem is pointed out and attention is then restricted to the case of hyperelliptic curves. For this case explicit constructions for the variety N and the mapping π are given.

When the characteristic of k is different from 2, a hyperelliptic field K/k of genus g (>1) is defined by an equation, $Y^2=f(X)$, where $f(X)$ is of degree $m=2g+2$. The zeros of f on the line S_1 form a set $(P_1, P_2, \cdots, P_m)$ of m points associated with K/k. Two hyperelliptic curves are birationally equivalent if and only if their associated sets are projectively equivalent on S_1. The representation of K/k by the symmetric product $\bar{P}=P_1\otimes P_2\otimes\cdots\otimes P_m$ of its associated points provides the basic tool for the construction of the moduli-variety N. When $g>2$, it is shown that the variety N has the further property that the hyperelliptic fields K/k which admit more than one non-identical k-automorphism correspond to singular points of a derived normal model $\bar{N}$ of N. *H. T. Muhly.*

Alguneid, A. R. Analytical degeneration of complete twisted cubics. Proc. Cambridge Philos. Soc. **52** (1956), 202–208.

In its role as a geometric variable, the twisted cubic curve of S_3 was regarded by Schubert as a union $C=$

(C_L, C_E, C_T) of a point-locus C_L, an envelope of osculating planes C_E, and a system C_T of tangent lines, and he listed accordingly eleven first order degenerate forms of C with only slight indications as to how the existence of some of them might be formally established. If we obtain the generic C as the transform of a fixed non-singular C_0 by a generic (complete) collineation T, it appears that the eleven degenerations in question can all be obtained as proper specializations of C, either directly by substituting special values for the indeterminates occurring in the equations of T, or (in some cases) indirectly by selecting a suitable type of approach path to a degenerate complete collineation T' of some chosen type. The same method provides a formidable array of degeneration of C of higher orders; but some awkward structural features of this array suggest to the author that a more symmetrical result might be obtained by adjoining to C_L, C_E, C_T certain other aspects of a twisted cubic, namely its chord congruence C_C, its congruence C_A of axes, and its congruence C_0 of osculation lines. *J. G. Semple.*

Permutti, Rodolfo. Su una rappresentazione delle $g_n{}^1$ di una retta. Rend. Accad. Sci. Fis. Mat. Napoli (4) 22 (1955), 337–339 (1956).

Par l'intermédiaire d'une conique C, on fait correspondre à un couple de points d'une droite D, le pôle dans C de la droite joignant leurs homologues sur C; on peut ainsi représenter une correspondance symétrique sur la droite par une courbe algébrique du plan de C. Si l'on se donne une correspondance symétrique T_{n-1} d'ordre $n-1$ sur D, telle qu'il existe un groupe de n points distincts $G_n=P_1+P_2+\cdots+P_i+\cdots+P_n$ tel qu'à tout P_i, T_{n-1} fasse correspondre G_n-P_i, la courbe image de T_{n-1} est d'ordre $n-1$ passant par le groupe Q des intersections 2 à 2 des tangentes aux homologues des P_i; réciproquement une courbe d'ordre $n-1$ passant par Q représente une $g_n{}^1$ contenant G_n. Les courbes d'un tel système sont ou irréductibles ou formées d'une courbe irréductible et de tangentes à C. On en déduit une généralisation du théorème de Poncelet: Si une courbe d'ordre $n-1$ est inscrite à un n-latère circonscrit à une conique C, il y a une infinité de n-latéres circonscrits à C, inscrits à la courbe. Enfin on peut en tirer une démonstration simple du théorème de Lüroth. *B. d'Orgeval* (Dijon).

Thonet, Raymond. Sur les points de diramation des surfaces multiples d'ordre 23. Bull. Soc. Roy. Sci. Liège 25 (1956), 119–127.

Thonet, Raymond. Sur la structure de trois points de diramation de surfaces multiples. Acad. Roy. Belg. Bull. Cl. Sci. (5) 42 (1956), 328–355.

Etude par la méthode classique de Godeaux des points de diramation A' d'une surface image d'une involution cyclique d'ordre premier p, ne possédant qu'un nombre fini de points unis, appartenant à une surface algébrique, dans le cas où le point uni A correspondant étant de 2° espèce, les trois points intersections des 4 courbes rationnelles équivalentes au domaine de A' sont doubles coniques ou biplanaires; la première hypothèse est traitée pour $p=239$, $a=101$, $p=839$, $a=249$, la seconde pour $p=1021$, $a=451$. Dans le premier cas le point A' équivaut à 7 courbes rationnelles de degré virtuel: -3, -2, -3, -2, -3, -2, -4; il est d'ordre 7, son plan tangent se compose de: Un cône du 2° ordre, 2 plans un cône du 3° ordre, chacun rencontrant le précédent selon une droite sur laquelle se trouve un point double ordinaire. Dans le 2° cas, A' équivaut à 7 courbes rationnelles de degré virtuel: -4, -2, -3, -2, -4, -2, -6, il est d'ordre 11, son cône tangent est formé de: 1 cône du 3° ordre, 1 plan, 1 cône du 2° ordre, un cône du 5° ordre, chacun rencontrant le précédent selon une droite sur laquelle se trouve un point double ordinaire. Dans le 3° cas, A' équivaut à 10 courbes rationnelles de degré virtuel: -3, -2, -2, -3, -2, -2, -3, -2, -2, -7; il est d'ordre 10; son cône tangent est formé de 4 cônes rationneles: 1 d'ordre 2, 2 plans, un cône d'ordre 6, chacun rencontrant le précédent selon une droite sur laquelle se trouve un point double biplanaire. *B. d'Orgeval* (Dijon).

Leicht, J. Über die Mannigfaltigkeit der reduziblen quadratischen Formen. Monatsh. Math. 60 (1956), 123–129.

The set of coefficients of a quadratic form in $n+1$ indeterminates over a field k can be regarded as homogeneous coordinates of a point in projective space S_N where $N=\frac{1}{2}n(n+3)$. Those points which correspond to reducible forms form an irreducible variety V. If A is the ideal of V the author shows by direct computation that the Hilbert polynomial $H(t, A)$ of A is

$$\tfrac{1}{2}(C(n+t, n)+1)C(n+t, n).$$

H. T. Muhly (Iowa City, Ia.).

Stein, E. Product varieties of two rational normal curves. Nederl. Akad. Wetensch. Proc. Ser. A. 59=Indag. Math. 18 (1956), 104–107.

In this paper the author investigates the product varieties of two rational normal curves which are a special case of the mixed Segre-Veronese varieties. The order of these varieties is derived from the plane representation, and some new properties, the section genus and the projections of the varieties are discussed. Also models of correspondences between the two rational curves are determined. *M. Piazzolla Beloch* (Ferrara).

Gallarati, Dionisio. Sul contatto di superficie algebriche lungo curve. Ann. Mat. Pura Appl. (4) 38 (1955), 225–251.

B. Segre [Accad. Italia. Mem. Cl. Sci. Fis. Mat. Nat. 5 (1934), 479–576] showed that if two surfaces F, G in S_3, of orders m, n, touch along a curve $\mathcal{C}$ of order $\frac{1}{2}mn$, F must in general have some nodes on $\mathcal{C}$, and if the number of these is s, the genus of $\mathcal{C}$ is

$$p=1+\tfrac{1}{8}mn(2m+n-8)-\tfrac{1}{4}s$$

whence $s\equiv\frac{1}{2}mn(2m+n)$ (mod 4). The main result of the present paper is a partial converse of this: if F is of order $m\leq 4$, and has on it a curve $\mathcal{C}$ passing through s ordinary nodes of F, and of the order and genus given by the above formulae, for some $n\geq m$, there exist surfaces G of order n touching F along $\mathcal{C}$. This is preceded by some general study of the contact (of any order) of hypersurfaces in any space, and of the singular loci which each hypersurface must have on the contact locus; and followed by some attempts to extend the main result to the cases $m=5, 6$, which are admittedly incomplete owing to our comparative ignorance of the possible surfaces of these orders with given numbers of nodes, but which at any rate produce no counter examples. *P. Du Val* (London).

Wallace, Andrew H. On the homology theory of algebraic varieties. I. Ann. of Math. (2) 63 (1956), 248–271.

The object of this paper is to give a rigorous proof of the following theorem of Lefschetz: Let V be a non singular

variety of dimension r in projective space and let V_0 be a non singular hyperplane section of V; then, if $q \leqq r-1$, every q-cycle on V is homologous to one on V_0. As was already the case in the Lefschetz proof, V_0 is considered as a member of a pencil Π of hyperplane sections of V; it is assumed that the following conditions are satisfied: a) the base points of Π form a non singular variety P (if $r>2$); b) only a finite number of members of Π have singularities; c) the singularities of any member of Π (if any) consist of a single point not on P. The members of Π are parametrized by the points ζ of the Riemann sphere S; let $V(\zeta)$ be the member corresponding to ζ, and, for any subset A of S, let $V(A)$ be the union of the sets $V(\zeta)$ for $\zeta \in A$. There are two kinds of points at which the slicing of V by the sets $V(\zeta)$ presents irregularities: the points of P and the singular points of the $V(\zeta)$'s. In order to study the nature of the slicing in the neighbourhood of these singular points, the author makes a careful study of the orthogonal trajectories of certain one-parameter families of varieties $V(\zeta)$ (obtained by letting ζ run over a straight line segment on S).

Let the non singular section V_0 correspond to ζ_0; let ζ_i $(1 \leqq i \leqq h)$ be the values of ζ for which $V(\zeta)$ is singular, and let λ_i be the segment joining ζ_0 to ζ_i. Let K_0 be the union of the sets λ_i, and let $S=U \cup U_0$ be a covering of S by two open sets such that U does not meet K_0. The problem is to prove that $H_q(V, V_0)=\{0\}$. This problem is divided into two parts, in such a way as to separate the roles played by the two kinds of irregular points. First it is proved that $H_q(V, V(U_0))=\{0\}$ provided $H_{q-2}(V_0, P)=\{0\}$ (the condition will be satisfied for $q \leqq r-1$ because the proof is made by induction on r and P is a hyperplane section of the $(r-1)$-dimensional variety V_0). In order to accomplish this, it is first proved that $H_q(V, V(U_0))$ is isomorphic to $H_q(V(U), V(U \cap U_0))$ (this already necessitates a study of a neighbourhood of P, the excision axiom net being directly applicable to this situation). Over U, the only irregularities of the slicing occur at the points of P; it is shown that $V(U)$ is the image of a fiber bundle X over U with fiber V_0 under a continuous mapping which maps upon P a subset of X which is homeomorphic to $U \times P$. From the consideration of this bundle, the author deduces that

$$H_q(V(U), V(U \cap U_0))=\{0\}.$$

From the equality $H_q(V, V(U_0))=\{0\}$, the author passes to $H_q(V, K_0)=\{0\}$, and it follows that $H_q(V, V_0)$ is generated by the images of the groups $H_q(V(\lambda_i), V_0)$ under the injection maps. Taking these groups one at a time, one may assume that ζ_0 is as near as one wants to one of the points ζ_i, and one is reduced to the study of a q-cycle ω modulo V_0 on $V(\lambda_i)$ in some neighbourhood of the unique singular point C of $V(\zeta_i)$. In order to study such cycles, the author introduces a new pencil Π' whose member V_0' through C is non singular; to ω he associates a cycle ω' modulo V_0' which is homologous to 0 modulo V_0' because ω' is carried by a small neighbourhood of the non singular V_0'; then one is able to come back to ω and to prove that ω is homologous to 0 modulo V_0 by using the pencil determined by V_0 and V_0'.

Although the aim of the paper is to give a complete and rigorous proof of a previously known theorem, a number of statements in his paper are substantiated by such assertions as "it is not hard to see that" or similar remarks; these gaps can only be filled by someone thoroughly conversant with the fine points of the technique of homology theory. *C. Chevalley* (Paris).

Chow, Wei-Liang. Algebraic varieties with rational dissections. Proc. Nat. Acad. Sci. U.S.A. 42 (1956), 116–119.

Let V be an abstract algebraic variety, and let $\{W_i\}$ be a set of varieties contained in V such that every point of V is contained in one and only one of these varieties. If the set $\{W_i\}$ is finite, such a set is called a dissection of V. This dissection is said to be rational if each variety W_i is equivalent to an affine space by an everywhere biregular birational transformation. An element W in a rational dissection is called a cell, or an r-cell if r is the dimension of W.

With suitable definitions the following theorem is proved. The set of all s-cells in a rational dissection of a variety V constitute a rational base for the s-cycles in V, which is defined over any field of definition for the rational dissection. This theorem generalises the basis theorem for Grassmann varieties proved in Hodge and Pedoe, "Methods of algebraic geometry" [v. 2, Cambridge, 1952, p. 337; MR 13, 972]. *D. Pedoe* (Khartoum).

Gallarati, Dionisio. Sulle superficie algebriche dello spazio ordinario che osculano lungo una curva una superficie cubica non rigata. Rend. Mat. e Appl. (5) 14 (1955), 674–685.

If two surfaces F, G of S_3, of orders m, n osculate each other along a curve C of order $\frac{1}{2}mn$, and F has s binodes and s' triple points at simple points of C, the genus of C is

$$p=1+\tfrac{1}{12}mn(3m+n-12)-\tfrac{1}{3}s-\tfrac{2}{3}s',$$

whence

$$2(s+2s') \equiv \tfrac{1}{4}mn^2 \pmod 3.$$

This is applied to the case $m=3$ (F non-ruled), and it is found that C is either itself a complete intersection, or is of order $3k$ and genus $\frac{3}{2}k(k-1)$ passing through three binodes, or of order $3k+1$ and genus $\frac{1}{2}k(3k-1)$ or order $3k+2$ and genus $\frac{1}{2}k(3k+1)$ passing through two binodes. It is further shown that if F has three binodes B_3 or a unode U_8, every curve on F is a locus of osculation with another surface, and counted three times forms the whole intersection of the two. No other cubic surface has this property. *P. DuVal* (London).

Gutwirth, A. Classification crémonienne de deux classes de plans multiples cycliques rationnels. Technion. Israel Inst. Tech. Sci. Publ. 6 (1954/5), 38–57. (Hebrew summary)

It is shown by applying a sequence of quadratic transformations to the corresponding n-ple plane, that every plane involution generated by a cyclic de Jonquières transformation of order n is a Cremona transform of that generated by the cyclic homology

$$x'=\varepsilon x, \quad y'=y, \quad z'=z \quad (\varepsilon^n=1).$$

It is further shown that the cyclic plane involutions of order 6 reduce to only five general types, the other six types given by Bottari [Ann. Mat. Pura Appl. (3) 2 (1899); 277–296; Giorn. Mat. Battaglini (2) 10(41) (1903), 285–320] being Cremona transforms of special cases of these. *P. DuVal* (London).

Marchionna, Ermanno. Sulle varietà multiple non lineari: estensioni del teorema d'Enriques relativo all'esistenza dei piani multipli. Ann. Mat. Pura Appl. (4) 38 (1955), 321–338.

On a surface F in S_3 a curve D is given; the problem is the construction of multiple surfaces covering F, with D

as branch curve. The author considers a pencil of plane sections, say $y=\lambda x$, and the multiple plane obtained by projecting F from a point on the axis the pencil, say that at infinity; the branch curve of this multiple plane (apparent contour of F) is denoted by Φ. If a general curve f of the pencil is the seat of a μ-ple curve, branching at its intersections with D, a necessary and sufficient condition for this to be the section of a μ-ple F branching on D is the following: For each curve of the pencil for which two of the branch points of the multiple line, projection of the μ-ple (not the simple) f, coincide in a point H, the permutations a, b defined on a double line of the pencil near the critical one by cycles about the points A, B which coincide in the limit, satisfy $ab=1$, $(ab)^2=1$, $(ab)^3=1$, $(ab)^4=1$ according as H is a simple point, a node, a cusp, or a tacnode, of the total curve $\Phi+D'$ (D' being the projection of D) [In the special case in which F is a plane the result is due to Enriques, Ann. Mat. Pura Appl. (4) 1 (1924), 185–198.]

P. DuVal (London).

Hodge, W. V. D. A note on the Riemann-Roch theorem. J. London Math. Soc. 30 (1955), 291–296.

The author notes that by the method of sheaves in algebraic geometry an exact formula for the dimension of a complete linear system D of primals on a non-singular variety V has been given [D. C. Spencer, Proc. Nat. Acad. Sci. U.S.A. 39 (1953), 660–669; MR 16, 618]. He exhibits this formula expressed in a manner familiar to classical algebraic geometry. The author gives the formula in terms of certain characters of D.

Let Ω^r denote the sheaf of germs of holomorphic r-forms on V, and let $\Omega^r(D)$ denote the sheaf of germs of meromorphic r-forms which are multiples of $-D$. By K the canonical divisor is denoted. The author sets $\sigma_V{}^r(D)=\dim H^r(V, \Omega^0(D))$, and then gives $\sigma_V{}^r$ the following geometrical interpretation; namely, if $|X_1|, \cdots, |X_r|$ are r ample systems such that $|X_i+D-K|$ is ample for $i=1, \cdots, r$, then the deficiency of the system cut by $|D+X_1+\cdots+X_r|$ on $X_1\cap\cdots\cap X_r$ is equal to $\sigma_V{}^r(D)$, and hence is independent of the systems $|X_1|, \cdots, |X_r|$ chosen.

The number $\sigma_V{}^r$ is the rth index of irregularity of D on V.

P. E. Conner (Princeton, N.J.).

Guggenheimer, H. L'invariance des genres des surfaces complexes. Bull. Res. Council Israel Sect. A. 5 (1955), 26–27.

Indications concernant l'extension de la notion de genre à une variété à structure complexe S sans singularités. On utilise le groupe de cohomologie des formes differentielles, de type (r, s), relativement à la differentielle d'', soit $H_{r,s}''(S)$; ou pose $p_g=\dim H_{2,0}''(S)$, $p_a=p_g-\dim H_{1,0}''(S)$ pour obtenir une extension à S du genre géométrique et du genre arithmétique; on indique que ces notions sont invariantes pour des modifications qui se ramènent à une succession de modifications ponctuelles (éclatements de Hopf).

P. Lelong (Paris).

Guggenheimer, H. Sur la signature des variétés algébriques. Bull. Res. Council Israel Sect. A. 5 (1955), 23–25.

M et M' étant variétés algébraiques sans singularités, en correspondance birationnelle, $V\subset M$ et $V'\subset M'$ étant les sous-variétés des éléments exceptionnels dans la correspoudance, les indices d'inertie des variétés satisfont à la relation

$$\tau(M)-\tau(V)=\tau(M')-\tau(V').$$

Expression de $\tau(M)$ à partir de p_{rs}, rang du groupe des formes fermées par celui des formes $\alpha_{r,s}$ pour lesquelles $\alpha_{r,s}=d'd''\alpha_{r-1,s-1}$.

P. Lelong (Paris).

Derwidué, L. Sur le comportement associé et la réduction des singularités. Bull. Soc. Roy. Sci. Liège 24 (1955), 212–238.

The concept in relation to d-folds on an irreducible algebraic $(d+1)$-fold, of "associated behaviour" of one d-fold W with another d-fold V at a point P common to both, was first introduced by B. Segre [Ann. Mat. Pura Appl. (4) 33 (1952), 5–48; MR 14, 683] and then figured prominently in Derwidué's work [Mém. Soc. Roy. Sci. Liège (4) 13 (1953), 1–41; MR 15, 551]. In the present paper Derwidué returns to formulate and develop his own account of Segre's idea, and to examine more critically the ways in which he proposes to use it. If the equations

$$(1) \qquad F_\alpha(x_0, \cdots, x_N)=0 \quad (\alpha=1, \cdots, s)$$

define an irreducible algebraic variety V of dimension d in a projective space S_N over the complex field k, we can define the "generic form" of the equations of V to be

$$(2) \qquad G_\alpha(y_0, \cdots, y_N)=0 \quad (\alpha=1, \cdots, s),$$

these equations being derived from (1) by the substitution $x_i=\sum_{j=0}^N a_{ij}y_j$ $(i=0, \cdots, N)$ in which the a_{ij} are all independent indeterminants. Further, the "characteristic equation" of V, obtained by eliminating $y_{d+2}, \cdots, y_N$ from (2), is

$$(3) \qquad \mathcal{V}(y_0, \cdots, y_{d+1})=0,$$

where $\mathcal{V}$ is a form with coefficients in $k(a_{ij})$; and this equation represents the cone $\mathcal{V}(a)$ projecting V from the generic $[N-d-2]$ given by $y_0=\cdots=y_{d+1}=0$. This definition can be extended to pure d-folds of S_N. Suppose then that W is another pure d-fold of S_N, with characteristic equation $\mathcal{W}(y_0, \cdots, y_{d+1})=0$. We say that W has "associated behaviour of the first kind" with V at a common point P is there exists an identity

$$\beta\mathcal{W}=\alpha_0\frac{\partial\mathcal{V}}{\partial y_0}+\cdots+\alpha_{d+1}\frac{\partial\mathcal{V}}{\partial y_{d+1}},$$

where $\beta, \alpha_0, \cdots \alpha_{d+1}\in k(a_{ij})[y_0, \cdots, y_{d+1}]$ and β does not vanish at P. There is again an obvious extension of this definition to "association of the ith kind" of W with V at P. The important case is that in which V and W are immersed in an irreducible $(d+1)$-fold U of S_N, while P is a simple point of U; for this is the case in which associated behaviour is a useful generalization of the behaviour of the polar manifolds of V on U at singular points of V. Every application, in fact, starts with a system of such polar manifolds of V on U, the effect of dilatations on their relation to V having then to be studied.

The main results of the paper are the above basis for a formal theory of associated behaviour, extensions of the concept, in particular to circumstances in which V or U or both vary in irreducible algebraic systems in S_N, an algebraic discussion of the effects of dilatations on associated behaviour, a further investigation of certain critical steps in the author's previous paper (already quoted), and finally a revised version of his proof of the reduction theorem for algebraic surfaces. The reviewer has objec-

tions to some of the language in the paper, as for example when it is suggested that any particular member of an algebraic system is to be defined always by giving "fixed values" to the "independent indeterminate parameters" occurring in the definition of the generic member. Also assertions are frequently made for which explanations and proofs should be provided. For basic procedures, there are many references to the author's recent preparatory memoir in Bull. Soc. Roy. Sci. Liège **24** (1955), 172–188 [MR **17**, 6]. *J. G. Semple* (London).

Denniston, Ralph H. F. Sui numeri di Betti delle varietà razionali. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 418–421 (1956).

It is known that rational curves and surfaces without singular points are without transcendental homology-classes. It has been conjectured by Ehresmann [Ann. of Math. (2) **35** (1934), 396–443, p. 443] that this theorem remains true for rational algebraic varieties without singular points, whatever the dimension of the variety. The author constructs effective counterexamples for which the Betti-numbers corresponding to odd dimensions are not zero, so that transcendental homology-classes must exist. Use is made of the work of B. Segre on dilatations [Ann. Mat. Pura Appl. (4) **33** (1952), 5–48; MR **14**, 683]. *D. Pedoe* (Khartoum).

Gutwirth, A. Systèmes canoniques et pluricanoniques des plans multiples abeliens d'ordre p^2 et cycliques d'ordre pq. Rend. Mat. e Appl. (5) **13** (1955), 440–471.

Let a surface F have on it effective canonical or pluricanonical systems, and be transformed into itself by an abelian group T of order p^2 (p prime), generating a rational involution represented on a p^2-ple plane ϖ. The branch curve of ϖ consists of $p+1$ parts, each of which is the image of points fixed for a subgroup T_i ($i=1, \cdots, p+1$) say of order p in T; common points of any two of these are common to all and represent fixed points for the whole group T. There are $p+1$ p-ple planes, each representing the mapping on ϖ of a surface F_i, image of the involution generated by a subgroup T_i, and the branch curve, B_i of such a p-ple plane contains all the parts of the total branch curve except that corresponding to T_i. B_i is a linear combination of irreducible curves, with coefficients which are residues mod p, and kB_i ($k=1, \cdots, p-1$) is equally a branch curve for the same multiple plane, since the surfaces

$$z^p=f(x, y), \quad z^p=[f(x, y)]^k$$

(after removal of any p-fold factor as the right) are birationally equivalent.

The canonical system, and each pluricanonical system, on F has p^2 linear subsystems, actual or virtual, compounded with T; each has some part of the united curve as fixed part, and the variable part appears on ϖ as a linear system; the difference between any two of these systems on ϖ is a virtual pth part of one of the curves B_i. Expressions for all these systems are given in great detail, but are not easy to follow on account of the overbrief explanations of the notations.

In the second part of the paper similar investigations are made for the case where the group T is cyclic of order pq (p, q both prime). *P. Du Val* (London).

Northcott, D. G. On the algebraic foundations of the theory of local dilatations. Proc. London Math. Soc. (3) **6** (1956), 267–285.

Étude algebrique locale des transformations birationnelles appelées „dilatations" par B. Segre [Ann. Mat. Pura Appl. (4) **33** (1952), 5–48; MR **14**, 683]. Soit V^d une variété irréductible sur un corps algébriquement clos K, et soit (x) un point générique de V sur K; on suppose que l'origine O est un point simple de V; soit M^r ($0 \leqq r \leqq d-2$) une sous variété de V admettant O pour point simple. On choisit un entier t suffisamment grand et on considère un système de générateurs $(m_j(X))$ de l'espace vectoriel des polynômes de degré $\leqq t$ qui sont nuls sur M; le lieu V' du point $(m_j(x))$ sur K est appelé une transformée de V par dilatation locale de M par rapport à O. La transformée L' de O par cette dilatation est une sous variété linéaire de dimension $d-r-1$ de V'; la transformée M' de M est une sous variété de dimension $d-1$ de V'. Un préliminaire algébrique montre comment les anneaux locaux $\mathfrak{o}(L'; V')$ et $\mathfrak{o}(M'; V')$ se déduisent intrinsèquement de $\mathfrak{o}(O; V)$ et l'idéal $\mathfrak{p}$ de M dans $\mathfrak{o}(O; V)$. La transformation de dilatation est birégulière pour les diviseurs W de V; si O' est un point du transformé W' de W qui correspond à O, on peut choisir des systèmes réguliers de paramètres $(s_1, \cdots, s_d)$ de $\mathfrak{o}(O; V)$ et $(t_1, \cdots, t_d)$ de $\mathfrak{o}(O'; V')$ tels que $s_i=t_i$ pour $i=1, \cdots, r$, d et $s_j=t_jt_d$ pour $j=r+1, \cdots, d-1$; alors, si $F(s_1, \cdots, s_d)=0$ est une équation analytique locale de W en O, $F(t_1, \cdots, t_r, t_{r+1}t_d, \cdots, t_{d-1}t_d, t_d)=0$ est une équation analytique locale de W' en O'. *P. Samuel* (Clermont-Ferrand).

Abhyankar, Shreeram. Local uniformization on algebraic surfaces over ground fields of characteristic $p\neq 0$. Ann. of Math. (2) **63** (1956), 491–526.

Soit K un corps de fonctions algébriques à deux variables sur un corps algébriquement clos k. Une valuation v de K/k est dite uniformisable s'il existe un modèle V de K sur lequel le centre de v soit simple. Le théorème d'uniformisation locale est l'assertion que toute valuation v de dimension 0 de K/k est uniformisable (l'assertion correspondante pour les valuations de rang 1 est alors presque triviale, soit au moyen du théorème d'uniformisation locale, soit grâce à la méthode de normalisation). Ce théorème a été démontré par Zariski lorsque K est de caractéristique 0 [Ann. of Math. (2) **40** (1939), 639–689; MR **1**, 26]. L'auteur le démontre ici pour un corps K de caractéristique $p\neq 0$. Les méthodes employées par Zariski lorsque v est de rang 2, ou discrète de rang 1, s'appliquent ici sans modification. Dans le § 1 on montre que la méthode de Zariski légèrement modifiée s'applique au cas où v est irrationnelle de rang 1 (c'est à dire où son groupe des valeurs est de la forme $Z+Zt$, t étant irrationnel). Reste donc le cas où le groupe des valeurs de v est un sous groupe dense du groupe additif des nombres rationnels. Pour traiter ce cas l'auteur utilise une méthode de „montée et descente": il considère une extension transcendante pure $K_1=k(x, y)$ contenue dans K et sur laquelle K est algébrique séparable, la plus petite extension galoisienne K^0 de K_1 contenant K, et les groups et corps de décomposition et d'inertie (relatifs à certain anneaux locaux) des extensions galoisiennes K^0/K_1 et K^0/K. Il se ramène ainsi à démontrer que, si L^0/L est une extension cyclique de degré premier q et w une valuation rationnelle de L qui admette une seule extension w^0 à L^0, alors, pour que w^0 soit uniformisable, il faut et il suffit que w le soit (§ 4). Pour $q\neq p$ la montée est facile (§ 5), et la descente utilise une suite de transformations quadratiques (§ 6). La démonstration du théorème de montée pour $q=p$ est la plus délicate: on se ramène d'abord à une extension de degré p du corps des fractions de l'anneau de séries formelles complété de l'anneau local régulier du centre de w

sur un modèle convenable de L (§ 7), puis on effectue des transformations quadratiques convenables de cet anneau de séries formelles (§§ 8 et 9).

Un § préliminaire (§ 2) démontre ou rappelle les résultats les plus utiles de la théorie de Galois des anneaux locaux (groupes de décomposition et d'inertie). Un autre (§ 3) donne quelques lemmes sur les transformations quadratiques.

Rappelons que la validité du théorème d'uniformisation locale pour les surfaces en caractéristique $p\neq 0$ permet maintenant d'étendre immédiatement à la caractéristique $p\neq 0$ d'importants résultats démontrés par Zariski et Muhly en caractéristique 0 [ibid. **43** (1942), 583–593; Bull. Sci. Math. (2) **78** (1954), 155–168; Trans. Amer. Math. Soc. **69** (1950), 78–88; MR **4**, 52; **16**, 398; **12**, 278].

P. Samuel (Clermont-Ferrand).

Broeckx, Raymond. Théorèmes généraux avec des méthodes de preuves sur la théorie des intersections des courbes algébriques d'après L. Casteels. Meded. Kon. Vlaamse Acad. Kl. Wetensch. **16** (1954), no. 4, 15 pp. (Dutch. French summary)

Let A_n be a curve of degree n in a projective plane P. We say that the n-rank of a set of points in P is r when the system of linear equations in the coefficients of A_n which is obtained by substituting these points into $A_n=0$ has rank r.

The paper and also the two papers reviewed below are based on the methods of Casteels [Verh. Vlaamsche Acad. Wetensch., Lett. Schone Kunst. België **15** (1953), no. 41; MR **16**, 951]. Of the propositions given the following is a typical one. Let $p\leqq n$, $q\leqq n$, $p+q>n$, let B_p and C_q have no irreducible components of multiplicity >1 and let D_s intersect C_q in sq distinct points not on B_p. Then if y common points of B_p and C_q have n-rank $y-x$ the set of $sq+y$ points has $(n+s)$-rank $y+sq-x$.

F. J. Terpstra (Pretoria).

Casteels, L. Le théorème des $3n-1$ points dans la construction d'une A_n. Meded. Kon. Vlaamse Acad. Kl. Wetensch. **16** (1954), no. 16, 11 pp. (Dutch. French summary)

The folowing theorem is shown [for notation and reference see the preceding review]. The n-rank of $3n-1$ given points is $3n-1$ if and only if no line contains $n+2$ of the given points and no conic contains $2n+2$ of them.

F. J. Terpstra (Pretoria).

Casteels, L.; et Broeckx, Raymond. Le théorème général des $\Phi(n, q)=q\ (n-q+3)-1$ points dans la construction d'une A_n, avec les groupes K et Φ liés avec le variant Φ. Meded. Kon. Vlaamse Acad. Kl. Wetensch. **17** (1955), no. 4, 35 pp. (Dutch. French summary)

When $n\geqq 2q-3$, $n\geqq q$ and $\Phi(n, q)=q(n-q+3)-1$ it is shown [for notation and reference see the second preceding review] that x given points with $x\leqq\Phi(n, q)$ have n-rank x if there exists no C_i $(i=1, 2, \cdots, q-1)$ which passes through $\Phi(n, i)+1$ of the given x points.

F. J. Terpstra (Pretoria).

Gherardelli, Francesco. Alcune osservazioni sulle varietà di Wirtinger. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **89** (1954–55), 387–400.

Riassunto d'autore: Si determinano i principali caratteri aritmetici e geometrici delle varietà di Wirtinger e si costruisce un esempio di superficie regolare avente biirregolarità positiva.

P. Roquette (Hamburg).

★ **Galafassi, Vittorio Emanuele. Classici e recenti sviluppi sulle superficie algebriche reali.** Colloque sur les questions de réalité en géométrie, Liège, 1955, pp. 131–147. Georges Thone, Liège; Masson & Cie, Paris, 1956. 250 fr. belges; 1900 fr. français.

An expository lecture on the general state of knowledge on questions of reality for algebraic surfaces. The work of Comessatti (to whom ten references are given) is chiefly dealt with, but at the end the author gives some of his own results on "abstract ruled surfaces" (i.e. birational transforms of ruled surfaces), in particular that the number of real sheets of such a surface (of any given genus) is unlimited, as is the order of connexion (or Euler characteristic) of each sheet.

P. Du Val (London).

★ **Brusotti, Luigi. Su talune questioni di realità nei loro metodi, risultati e problemi.** Colloque sur les questions de réalité en géométrie, Liège, 1955, pp. 105–129. Georges Thone, Liège; Masson & Cie, Paris, 1956. 250 fr. belges; 1900 fr. français.

This is an outline survey, with copious references to the very extensive bibliography (occupying ten pages), of the methods used and results which have been obtained on the reality properties of algebraic curves, especially the existence of curves (in the plane or in higher space) with a given number of real circuits, and of pencils with given numbers of real base points and "critical points" (nodes of members of the pencil). *P. Du Val* (London).

Godeaux, Lucien. Structure des points de diramation des surfaces multiples. Publ. Sci. Univ. Alger. Sér. A. **1** (1954), 223–238 (1955).

The author summarises briefly here, with references to earlier papers, his results on the isolated united points of a cyclic involution I of prime order on an algebraic suface F and the corresponding singularity A on a surface Φ, image of a system on F compounded with I and free from base points. In particular he shows how the numerical characters of the united point can be obtained from the analysis of the neighbourhoods of A, and gives an example in which the first neighbourhood of A consists of four lines meeting consecutively, the first two in a binode B_3, the second and third in a simple point, and the third and fourth in a binode B_2; this leads to an involution of order 79. *P. Du Val* (London).

Godeaux, Lucien. Structure de quelques points de diramation de surfaces multiples cycliques. I, II. Bull. Soc. Roy. Sci. Liège **24** (1955), 303–312; **25** (1956), 5–13.

In these two notes the author studies particular cases of an isolated fixed point of a cyclic involution on a surface. The first is an involution of order 31, which in the neighbourhood of the united point A has the form

$$x_0{}^1:x_1{}^1:x_2{}^1=x_0:\varepsilon x_1:\varepsilon^{22}x_2\ \ (\varepsilon^{31}=1),$$

where x_0, x_1, x_2 are homogeneous coordinates in the tangent plane at the point $A(1, 0, 0)$. The corresponding point A^1 on a surface image of the involution, is quadruple, the tangent cone breaking up into a quadric cone and two planes; the two planes only meet at A^1, but each has a line in common with the quadric cone; consecutive to A^1 along one of these lines is a conic node C_2 and along the other an ordinary binode B_3.

In the second example the involution is of order 41,

and the equations in the neighbourhood of A are

$$x_0^1 : x_1^1 : x_3^1 = x_0 : \varepsilon x_1 : \varepsilon^{35} x_2 \quad (\varepsilon^{41} = 1).$$

The tangent cone at A^1, which is triple on Φ, consists of three planes, two of which meet only at A^1 but each meets the third in a line; consecutive to A^1 along these lines are binodes B_5, B_4 respectively, i.e. the latter having a conic node C_2 and the former an ordinary binode B_3 consecutive to it in turn.

P. Du Val (London).

Rosina, B. A. Ulteriori sviluppi della teoria diametrale delle superficie algebriche. Ann. Univ. Ferrara. Sez. VII. (N.S.) 4 (1955), 51–67.

A diametral plane of an algebraic surface is the polar plane of a point at infinity, whose direction is called conjugate to the plane. General forms are given for the Cartesian equation of a surface having a "principal" point or line (either ordinary or at infinity), i.e. one through which all diametral planes pass. Principal diametral planes are also sought, i.e. those which are perpendicular to the conjugate direction. The methods are elementary and the results obvious, except where they are obviously wrong. It is stated in footnotes that a quadric with a principal line, whether ordinary or at infinity, is a paraboloid, whereas it is clearly a cylinder, general or parabolic respectively. A more serious blunder is the statement that a general surface of order n has n^2 principal diametral planes, which would give four to a general quadric! The correct number is n^2-n+1.

P. Du Val (London).

Marchionna, Ermanno. Sopra una disuguaglianza fra i caratteri proiettivi di una superficie algebrica. Boll. Un. Mat. Ital. (3) 10 (1955), 478–480.

Given on an algebraic surface F a linear pencil of curves of genus π, with n base points, and including δ curves of genus $\pi-1$, topological considerations are used to show that $\delta \geqq n-1$, and that a necessary condition for equality is that $\pi=0$, whence F is rational.

A corollary is that the only non-singular surfaces in any space (or surfaces in S_3 with only ordinary singularities) whose class is less than their order are the plane and the Veronese surface, and the general projections of the latter into S_4 and S_3. *P. Du Val* (London).

Skopec, Z. A. Some methods of obtaining special Cremona transformations. Moskov. Gos. Univ. Uč. Zap. 155, Mat. 5 (1952), 73–93. (Russian)

It is shown that $U(x)=0$ being a curve in R_2, the equations $y_i=\partial U:\partial x_i$ define a Cremona transformation if and only if U degenerates into 3 lines or into a conic and one of its tangents. The author continues with a discussion of monoidal transformations in R_3 of the type

$$y_1 : y_2 : y_3 : y_4 = f_1 x_4 : f_2 x_4 : f_3 x_4 : f_4,$$

where $z_1:z_2:z_3=f_1:f_2:f_3$ is a Cremona transformation in R_2 of degree $n-1$ and f_4 a form in x_1, x_2, x_3 of degree n. Finally Cremona transformations are found which can be defined by a system of linear equations in y whose coefficients are linear or quadratic in x. All results are generalised to higher dimensional spaces. *F. J. Terpstra.*

See also: Chevalley, p. 1046; Godeaux, p. 1126; Guggenheimer, p. 1127.

NUMERICAL ANALYSIS

Frame, J. S. A continued fraction for periodic rent, logarithms, and roots. Pi Mu Epsilon J. 2 (1956), 176–183.

Several computational problems can be solved as special cases of the continued fraction expansion

$$f_1(x,y) = 1 + \frac{x^2-y^2}{3} + \frac{x^2-4y^2}{5} + \frac{x^2-9y^2}{f_4}.$$

It is shown that the value of this continued fraction is related to the periodic rent of an annuity. Also continued fractions for $\log_e(1+r)$, e^x, $x \cot x$, and $(a^m+b)^{1/m}$ are derived therefrom. *E. Frank* (Chicago, Ill.).

Hammersley, J. M. Conditional Monte Carlo. J. Assoc. Comput. Mach. 3 (1956), 73–76.

Conditional Monte Carlo applies when one wishes to estimate the expected value of a function $\varphi(z)$, where the vector z is a sample from a known $h(z)$ but the sample must satisfy some specified condition $\eta(z)=y_0$. The technique described, and illustrated, enables one to sample from the unconditional distribution and to find the expected value of a certain other function.

The basic formulas are simple. If X, Y, Z are spaces and J is the Jacobian of a (1, 1)-transformation between Z and the product space of X and Y, and if

$$G(x) = \int_Y g(x,y)\,dy,$$

then subject to mild conditions

$$\int_X \varphi(x) f(x)\,dx = \int_Z \varphi[\xi(z)]\omega(z)h(z)\,dz,$$

where

$$\omega(z) = f[\xi(z)]g[\xi(z), \eta(z)]J(z)/\{G[\xi(z)]h(z)\}.$$

A. S. Householder (Oak Ridge, Tenn.).

Forsythe, George E. Computing constrained minima with Lagrange multipliers. J. Soc. Indust. Appl. Math. 3 (1955), 173–178.

This note is intended to illustrate the new aspect which even the simple processes of classical analysis take when examined by the numerical analyst. The author considers the application of numerical steepest-descent methods to Lagrange multiplier solutions of minimizing problems with constraints. An example is given to illustrate some of the difficulties connected with the method, and a criterion is given for the success of the method involving the positive-definiteness of a quadratic form. A modification of the method, involving steepest descents on the set of points satisfying the constraints, which eliminates some of the difficulties, is outlined. The criterion for success is refined to apply to the modified technique. *W. S. Loud* (Cambridge, Mass.).

Varoli, Giuseppe. Sul metodo di iterazione e la determinazione del tasso in alcuni problemi di matematica finanziaria. Ist. Mat. Finanziaria Univ. Studi Bologna no. 1 (1955), 28 pp.

The author gives a brief exposition of the method of iteration for solving non-linear equations, and then makes application to several equations arising in the mathematics of finance. *A. S. Householder.*

Broeckx, Rob. Une construction géométrique en relation avec la formule d'interpolation de Lagrange. Simon Stevin 30 (1955), 232–237. (Dutch. French summary)

L'auteur donne une construction du point d'intersection S de la parabole (P_n), déterminée par sa direction asymptotique OY et par $n+1$ points A_i, avec une droite s parallèle à la direction OY. Par les n points donnés $A_1, A_2, \cdots, A_n$ les droites $a_1, a_2, \cdots, a_n$ sont menées parallèles à OY. OX est choisi différent de OY. Du point A_0 les points $A_1, \cdots, A_n$ sont projetés sur la droite s en $S_1^{(1)}, S_2^{(1)}, \cdots, S_n^{(1)}$. On détermine les points d'intersection $A_1^{(1)}, A_2^{(1)}, \cdots, A_n^{(1)}$ des droites $a_1, a_2, \cdots, a_n$ avec les droites menées respectivement par les points $S_1^{(1)}, S_2^{(1)}, \cdots, S_n^{(1)}$ parallèles à OX. En partant de $A_1^{(1)}$ une construction analogue donne les points $A_2^{(2)}, \cdots, A_n^{(2)}$ sur les droites $a_2, \cdots, a_n$, etc. On finit par obtenir deux points $A_{n-1}^{(n-1)}$, $A_n^{(n-1)}$ appartenant respectivement aux droites a_{n-1}, a_n. La droite $A_{n-1}^{(n-1)}$, $A_n^{(n-1)}$ coupe s au point demandé S. L'auteur donne deux démonstrations analytiques et une démonstration géometrique. *S. C. van Veen* (Delft).

Quade, W. Zur Interpolationstheorie der reellen Funktionen. Z. Angew. Math. Mech. 35 (1955), 144–156. (English, French and Russian summaries)

The author considers interpolation and quadrature formulas suitable for numerical computations. *P. Erdős.*

Capecchi, Walfredo. Semplici procedimenti di interpolazione di una funzione esponenziale doppia (catenaria). Statistica, Bologna 15 (1955), 567–582.

[As usual in Italian, "interpolazione" means curve-fitting and not interpolation in the English sense.] Numerical procedures are described for fitting to empirical data a curve form $(a/2)(e^{bx}\pm e^{-bx})$ or the reciprocal of such a form. *T. N. E. Greville* (Washington, D.C.).

Garcia, Juan. Numerical tabulation of equations. Las Ciencias 17 (1952), 17–48 (5 plates). (Spanish)

The author discusses interpolation in the logarithm tables of trigonometric functions, and possible constructions of tables for some equations. *E. Frank.*

★**Dowgird, Zygmunt. Krakowiany i ich zastosowanie w mechanice budowli. [Cracovians and their application in structural mechanics.]** Państwowe Wydawnictwo Naukowe, Warszawa, 1956. 168 pp. zł. 18.

Cracovians are rectangular arrays of numbers which are added like matrices, but which are multiplied column-by-column. They were developed by T. Banachiewicz, apparently because column-by-column multiplication is easier in desk computing than row-by-column multiplication. The present work expounds the definitions, notations, and properties of cracovian theory, and their use in problems of linear algebra. The second chapter is devoted to the solution of systems of linear equations by various methods of triangular decomposition. In the third chapter are discussed simple iterative methods for solving linear systems and for computing eigenvalues. The exposition is elementary. There are problems from structural mechanics, and many numerical examples of orders up to 5 or 6. The examples are oriented towards desk computation.

The fact that cracovian multiplication is non-associative causes various strained notations, and appears to the reviewer to be an overwhelming impediment. Nevertheless, cracovians have a minority of enthusiastic supporters, largely but not exclusively in Poland. *G. E. Forsythe* (Los Angeles, Calif.).

Allen, D. W. Numerical solution of n linear equations in n unknowns, and the evaluation of nth order determinant (complex coefficients). J. Roy. Aero. Soc. 60 (1956), 350–353.

Computational layout for successive elimination, with checks. Very little explanation, and no justification, "since the theory demands lengthy exposition". The notation is not immediately clear. *A. S. Householder.*

Peres, Manuel. On solution of systems of simultaneous linear equations. Las Ciencias 17 (1952), 443–449. (Portuguese)

With a slight modification of the author's notation, let the equations be written $\sum a_{ij}x_j=\alpha_i$ $(i=1, 2, \cdots, r)$. There are many ways of expressing the x_j in terms of variables $x_1', \cdots, x_{r-1}'$ so that a particular equation, say the first, is satisfied identically. One such substitution is given by

$$a_{11}x_1=\alpha_1-a_{11}x_1',$$
$$a_{12}x_2=a_{11}x_1'-a_{12}x_2',$$
$$\cdots$$
$$a_{1r}x_r=a_{1r-1}x'_{r-1}.$$

When the substitution is made in the second equation, the $x_1', \cdots, x_{r-1}'$ can be expressed in terms of $x_1'', \cdots, x_{r-2}''$. Eventually one has a single equation in $x_1^{(r-1)}$. Some remarks are made on checking for blunders.

Unfortunately the method calls for $r(r+1)/2$ divisions by distinct divisors. Hence as it stands it seems totally unsuited for automatic computation.

A. S. Householder (Oak Ridge, Tenn.).

Stulen, F. B.; and Lehman, F. G. A method of solving inhomogeneous linear simultaneous equations. J. Math. Phys. 35 (1956), 123–126.

The method, no longer new, is that of applying a suitable polynomial in the matrix in order to speed convergence of the iteration. The most comprehensive discussion to date is that by Stiefel [Comment. Math. Helv. 29 (1955), 157–179; MR 17, 88]. Such a method was described by Gavurin [Uspehi Mat. Nauk (N.S.) 5 (1950), no. 3(37), 156–160; MR 12, 209], but the authors say the method has been in use for 8 years. *A. S. Householder.*

Blanc, Ch.; et Liniger, W. Erreurs de chute dans la résolution de systèmes algébriques linéaires. Comment. Math. Helv. 30 (1956), 257–264.

The authors assume that in applying the method of Choleski the computed quantities are augmented by independent random variables with zero mean and known range, and compute the means (which are zero) and the covariance matrices of each of the two sets of random variables

$$\eta_i=\zeta_i-x_i,$$
$$\varrho_i=\sum a_{ij}\,\zeta_j-c_i,$$

which are, respectively, the errors and the residues. Only terms linear in the variances of the initial variables are retained.

The authors assert that whereas one can construct systems for which the elementary rounding errors violate

their hypothesis, this is in practice unimportant. If m decimals are carried, the variances are proportional to 10^{-2m}; hence by solving repeatedly a system with known solution, the number of places varying from one time to the next, one can estimate the coefficients of 10^{-2m} and compare with the computed values. They present a 5th order system solved 5 times and come out with reasonable agreement.

Finally they discuss the case where the c_i are themselves stochastic variables, saying this applies to the difference equations which arise in the replacement of derivatives by differences in differential equations.

A. S. Householder (Oak Ridge, Tenn.).

Mendelsohn, N. S. Some elementary properties of ill conditioned matrices and linear equations. Amer. Math. Monthly **63** (1956), 285–295.

Theorems are proved which illustrate the phenomenon that an ill-conditioned matrix A may have a "left inverse" X and "right inverse" Y, such that XA and AY differ little from the unit matrix I, while AX and YA differ considerably from I. It is then important, when solving linear equations $Ax=B$ by $x=A^{-1}B$, to use X and not Y as approximate inverse of A, and when solving $x'A=B'$ by $x'=B'A^{-1}$, to use Y and not X for A^{-1}. Some methods produce a good X, some a good Y, and others, particularly for symmetric matrices, produce X and Y of equal accuracy. *L. Fox* (Teddington).

Masse, Jean-Léon. Recherches de méthodes de perturbation à convergence rapide et à programmation facile sur machines électroniques à cartes perforées. J. Rech. Centre Nat. Rech. Sci. **6** (1955), 393–403.

The problem is that of finding the eigenvalues and eigenfunctions of a perturbed system given those of the initial system. The actual discussion presupposes a reduction to Euclidean space, where one seeks the eigenvalues and vectors of a matrix Λ_0+P, where Λ_0 is diagonal and where the elements of P are sufficientlys mall and its diagonal null. The method discussed is actually that of Jahn [Quart. J. Mech. Appl. Math. **1** (1948), 131–144; MR **10**, 152] and Collar [ibid. **1** (1948), 145–148; MR **10**, 152]. *A. S. Householder* (Oak Ridge, Tenn.).

Miles, John W. A note on numerical differentiation. Quart. Appl. Math. **14** (1956), 97–101.

Faisant suite à une communication que j'avais présentée [Proc. Internat. Congress Math., 1954, Amsterdam, v. 2, Noordhoff, Groningen, 1954, p. 357], l'auteur indique que, dans la relation matricielle

$$f^{(m)}=D^m f$$

qui donne, pour un polynome de degré n, les dérivées mèmes de f aux points x_i en fonction des valeurs de f en ces points, la matrice D peut s'écrire $C^{-1}AC$, les matrices C et A pouvant être écrites explicitement.

J. Kuntzmann (Grenoble).

Urabe, Minoru; and Nise, Shigetoshi. A method of numerical integration of analytic differential equations. J. Sci. Hiroshima Univ. Ser. A. **19** (1955), 307–320.

The authors study quadrature error estimates of the form proposed by P. Davis [J. Rational Mech. Anal. **2** (1953), 303–313; MR **14**, 907] also P. Davis and P. Rabinowitz [Math. Tables Aids Comput. **8** (1954), 193–203; MR **16**, 404]: $|E_0(\phi)|\leq\sigma_{E_0}\|\phi\|$ for functions $\phi(z)$ regular in $|z-x_0|<\varrho$ and of class H^2 there. Here

$$\|\phi\|^2=\int|\phi|^2ds,\quad z=x_0+\varrho\exp(it),$$

$$E_0(\phi)=\int_{u_0}^{u_1}\phi(u)du-h_0\sum_{j=-1}^{N}a_j\phi(u_{-j}),\ 2\pi\sigma_{E_0}{}^2=\sum_{n=0}^{\infty}|E_0(u^n)|^2.$$

For fixed abscissas u_{-j}, the norm of E_0, σ_{E_0}, becomes a function of the weights a_j and may be minimized by solving a certain linear system. In this way a new quadrature formula may be obtained. The authors compare the minimal formulas with the traditional (interpolatory) formulas. Conditions are given which indicate when the new formulas are more accurate than the traditional ones and approximate expressions for the minimal weights are derived. A table is included which gives the minimal formulas and corresponding σ's for various configurations. The paper concludes with an application to the numerical integration of ordinary differential equations.

P. Davis (Washington, D.C.).

Luke, Yudell L. Simple formulas for the evaluation of some higher transcendental functions. J. Math. Phys. **34** (1956), 298–307.

This interesting account deals with what is more usually called the Euler-Maclaurin integration formula which converts an integral $\int_0^{nh} f(x)dx$ into a multiple of the sum of ordinates $h\sum_0^n f(rh)$ together with a correction term expressed as a series depending on successive derivatives of $f(x)$; this series must be curtailed for practical use, and a remainder term ignored, possibly after studying its negligibility. A modified form of the formula, using ordinates $f\{(r+\frac{1}{2})h\}$ at mid-interval is also discussed. The approach, by use of Fourier series, is less familiar than that used in most discussions and is illuminating, the remainder terms are discussed fairly fully.

The paper is rounded-off by application of the method to several specific examples: — Bessel functions $J_\nu(z)K_\nu(z)$, Struve functions $H_\nu(x)$, complete elliptic integrals, the error function and related functions. A useful discussion.

J. C. P. Miller (Cambridge, England).

Zadiraka, K. V. A majorant-minorant method of estimating the eigenvalues of a one-dimensional self-adjoint boundary problem of fourth order. Ukrain. Mat. Ž. **8** (1956), 12–25. (Russian)

The problem considered is $y^{IV}+(q-\lambda_r)y=0$ with boundary conditions of the form $y\sin\alpha+y'''\cos\alpha=y'\sin\beta+y''\cos\beta=0$ at each end point. The idea of the method was outlined by the author in a previous paper [Dopovidi Akad. Nauk Ukrain. RSR **1954**, 243–249; MR **16**, 824]. *A. S. Householder* (Oak Ridge, Tenn.).

Buchdahl, H. A. Über Approximationen der Thomas-Fermi-Funktion. Ann. Physik (6) **17** (1956), 238–241.

New approximations, involving only simple algebraic functions of the form $\{(1+ax)(1+bx)(1+cx)\}^{-1}$ are obtained for the solution of the Thomas-Fermi differential equation $y''=x^{-1/2}y^{3/2}$, with $y(0)=1$, $y(\infty)=0$. Tables are given of three such functions, of which the most accurate has $a=0.05727$, $b=0.1536$, $c=0.9288$, and deviates little from the true values throughout the entire range of x.

L. Fox (Teddington).

Fischbach, Joseph W. The numerical solution of non-linear differential equations by the method of steepest descent. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Memo. Rep. no. **646** (1953), 23 pp.

The differential equation written symbolically as $r(y)=0$ is solved in the range $a \leqq x \leqq b$ by a process of successive approximation in which a trial vector y_n is steadily improved, according to the scheme $y_{n+1}=y_n+\varepsilon_n\Delta_n$, the parameter ε and function Δ being chosen, according to stated rules, so that the positive definite form $\int_a^b r^2(y)dx$ is minimised. The use of finite-difference equations for derivatives and integrals reduces the problem to the solution of a set of non-linear algebraic equations. There is an illustrative example, for a simple non-linear first-order differential equation, and comments on the rate of convergence of the process. *L. Fox.*

Grossman, D. P. On the solution of the first boundary problem for elliptic equations by means of nets. Dokl. Akad. Nauk SSSR (N.S.) **106** (1956), 770–772. (Russian)

A note added at the end indicates that Young's paper [Trans. Amer. Math. Soc. **76** (1954), 92–111; MR **15**, 562] came to the author's attention after completion of his own. The equation considered here is

$$b\partial^2u/\partial x^2+c\partial^2u/\partial y^2+d\partial u/\partial x+e\partial u/\partial y-gu=f$$

with u vanishing on the boundary and $b>0$, $c>0$, $g\geqq 0$, the boundary being piece-wise smooth and the region simply connected. The theorems are to the effect that an ordering of the grid points exists such that the "Seidel" iteration converges at twice the rate of the "simple" iteration. There are no proofs. *A. S. Householder.*

Meyer zur Capellen, W. Harmonische Analyse bei der Kurbelschleife. Z. Angew. Math. Mech. **36** (1956), 151–152.

Gorn, Saul. An experiment in universal coding. Ballistic Research Laboratories, Aberdeen Proving Ground, Md., Rep. no. **953** (1955), 83 pp.

This report gives a detailed description of an automatic coding scheme to be used with modern computers, which permits programs to be written in a single "Universal Code" capable of being translated by the given computer into the specific code which it uses. Programs in "Universal Code" might therefore be used with any computer for which such a translating program exists. "Universal Code" has been applied to the two computers ORDVAC and EDVAC which are located at the Ballistic Research Laboratories, and a simple test problem prepared in "Universal Code" has been solved by both of them with similar results.

Since the non-interpretive method is used, in which the entire program is first translated into specific computer code before its instructions are obeyed, it is possible to achieve a speed of solution comparable to that which would have been attained had the problem been programmed directly for the computer. In "Universal Code" programs are formed naturally from blocks called subroutines which, in turn, may be formed from more elementary subroutines. Each block or subroutine appearing in a program performs a complete operation such as might be indicated in a block of a flow chart describing the program. Ultimate subroutines which cannot be broken down into more elementary subroutines are called atomic subroutines and correspond to fundamental operations carried out by very few computer instructions.

D. E. Muller (Urbana, Ill.).

See also: Brauer, p. 1044; Brauer and Laborde, p. 1044; Dedecker, p. 1047; Muller, p. 1093; Rogla, p. 1160; Schöbe, p. 1077; Zeragya, p. 1091.

Tables

★ **Smirnov, A. D. Tablicy funkciĭ Èĭri i special'nyh vyroždennyh gipergeometričeskih funkciĭ dlya asimptotičeskih rešeniĭ differencial'nyh uravneniĭ vtorogo poryadka. [Tables of Airy functions and of special confluent hypergeometric functions for asymptotic solutions of differential equations of second order.]** Izdat. Akad. Nauk SSSR, Moscow, 1955. 261 pp. (2 inserts). 28 rubles.

Asymptotic forms for the solutions of ordinary linear differential equations of the second order containing a large parameter and possessing a transition point or a singularity, are often obtained by a comparison of the given differential equation with a differential equation of one of the two forms

$$\text{(1)}\qquad \frac{d^2U}{ds^2}+s^\alpha U=0 \quad (\alpha>-2),$$

$$\text{(2)}\qquad \frac{d^2V}{ds^2}+\left[\frac{p_0}{2}\left(1-\frac{p_0}{2}\right)\frac{1}{s^2}+\frac{1}{s}\right]V=0.$$

The first equation is called in this volume the generalized Airy equation, the second, the degenerate hypergeometric equation. [Both can be transformed into Bessel's equation.] $U_1(s,\alpha)$ and $U_2(s,\alpha)$ is a fundamental system of solutions of (1), U_1 and $s^{-1}U_2$ are power series in $s^{\alpha+2}$ both reducing to unity when $s=0$. $V_1(s,p_0)$ and $V_2(s,p_0)$ is a fundamental system of solutions of (2), $s^{-p_0/2}V_1$ and $s^{-1+p_0/2}V_2$ are power series in s both reducing to unity for $s=0$. In the case $\alpha=1$ of (1), corresponding to a simple transition point, the integrals

$$V_{11}(s)=\int_0^s\left[U_1^2(t,1)-\frac{3^{1/3}\Gamma^2(2/3)}{2\pi t^{1/2}}\right]dt,$$

$$V_{12}(s)=\int_0^s\left[U_1(t,1)U_2(t,1)-\frac{1}{2(3t)^{1/2}}\right]dt,$$

$$V_{22}(s)=1+\int_0^s\left[U_2^2(t,1)-\frac{\Gamma^2(1/3)}{2.3^{1/3}\pi t^{1/2}}\right]dt,$$

occurring in the second approximations, are also introduced.

The Introduction to the volume under review gives the basic formulas for all of these functions, graphs for some of them, asymptotic forms (for large values of α) of certain solutions of the differential equations

$$\frac{d^2y}{dx^2}+[\lambda^2x^\alpha r(x)+q(x)]y=0,$$

$$\frac{d^2y}{dx^2}+\left[\lambda^2\frac{r(x)}{x}+\frac{p_0}{2}\left(1-\frac{p_0}{2}\right)\frac{1}{x^2}+\frac{q(x)}{x}\right]y=0,$$

in terms of $U(s,\alpha)$ and $V(s,p_0)$ respectively, and a brief account of the numerical tables which form the principal part of the volume.

5D or 5S tables of $U_i(s,1)$, $U_i'(s,1)$ with second differences, $i=1, 2$, are given for $s=-6(0.1)10$, and of $V_{ij}(s)$, $i, j=1, 2$ for $s=0(.01)10$.

4D tables of $U_i(s, \alpha)$ with first differences and of $U_i'(s, \alpha)$ are given for $i=1, 2$, $\alpha=\pm\frac{1}{4}, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm\frac{2}{3}, \pm\frac{3}{4}, \frac{5}{4}, \frac{4}{3}, \frac{3}{2}, \frac{5}{3}, \frac{7}{4}$, 2 and $s=0(.01)6$.

4D tables of $V_i(s, p_0)$ with first differences and of $V_i'(s, p_0)$ are given for $i=1, 2$, $p_0=.1(.1)1$, $s=0(.01)10$. The first five zeros of $V_i(s, p_0)$ are also tabulated.

The tables are designed for use in connection with the formulas obtained by Dorodnicyn [Uspehi Mat. Nauk (N.S.) **7** (1952), no. 6(52), 3–96; MR **14**, 876].

A. Erdélyi (Pasadena, Calif.).

Tricomi, Francesco G. Valori numerici di funzioni ortogonali di Laguerre. Atti Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. **90** (1955–56), 63–70.

This table gives

$$l_n(x)=e^{-x/2}\,L_n(x)=e^{-x/2}\sum_{m=0}^{n}\binom{n}{m}\frac{(-x)^m}{m!}$$

to 6*S* for $n=0(1)10$, $x=0.1(.1)1(.25)6(1)14(2)34$; the range was chosen to include all zeros of all the functions. It is an extension and revision of an earlier table of the author [same Atti **76** (1941), 288–316; MR **7**, 486] and was prepared on a desk calculator using the recurrence relation: $(n+1)l_{n+1}=(2n+1-x)l_n-nl_{n-1}$, a method known to be dangerous, so that thorough checking is necessary. This was done by first computing $\mathbf{L}_n(x)=n!L_n(x)$, which is integral for integral x, by a similar recurrence and, then obtaining $l_n(x)$ for these values. Intermediate values were obtained from these by use of addition — and bisection-formula for the $l_n(x)$. *John Todd* (Washington, D.C.).

Huckel, Vera. Tabulation of the f_λ functions which occur in the aerodynamic theory of oscillating wings in supersonic flow. NACA Tech. Note no. **3606** (1956), 59 pp.

Tables are given of the real and imaginary parts of the function

$$f_\lambda(M, \bar{\omega})=\int_0^1 u^\lambda e^{-i\bar{\omega}u}J_0\left(\frac{\bar{\omega}}{M}u\right)du,$$

for the following ranges of parameters: $\lambda=0(1)11$; $M=1.2(0.1)1.6$, 1.8, 2.0(0.5)4.0, 5.0; $k=0(0.005)0.150(0.010)0.200(0.025)0.350(0.050)1.000(0.010)2.000$, where

$$\bar{\omega}=2kM^2/(M^2-1).$$

Seven decimals are given throughout, and the last figure is expected to be correct within a unit. There is no comment about interpolation, even in the k-direction. A brief introduction lists the aerodynamic problems involving this function. *L. Fox* (Teddington).

Porter, R. J. On irregular negative determinants of exponent 9*n*. Math. Tables Aids Comput. **10** (1956), 22–25.

The determinant $D=b^2-ac$ of the quadratic form $ax^2+2bxy+cy^2$ is called irregular if the principal genus is non-cyclic. The number of classes divided by the highest period is called the exponent of D. The first 58 values of $-D$ with exponents $9n$ are listed, giving the complete table up to 150,000. Tables are given which reveal special properties and a method for squaring a class of forms is outlined. The discriminant -3299 seems to play an exceptional role in various respects. *O. Taussky-Todd*.

Rahman, Anees. Two-centre integrals arising out of 2*s* and 2*p* atomic functions. Acad. Roy. Belg. Cl. Sci. Mém. Coll. in 4° (2) **14** (1955), no. 2, 13 pp.

"A great number of such integrals that arise··· [in connection with diatomic molecules]··· have been tabulated by Kopineck [Z. Naturf. **5a** (1950), 420–431; **6a** (1951), 177–183; **7a** (1952), 785–800; MR **12**, 410; **13**, 130; **14**, 643], the following being simply certain very useful combinations of Kopineck's integrals; the reason for having this tabulation is that the integrals tabulated here appear directly as elements of the determinantal equations which give the minimised energy". (From the author's introduction.) *A. Erdélyi* (Pasadena, Calif.).

Mathematical Machines

★**Moore, Edward F. Gedanken-experiments on sequential machines.** Automata studies, pp. 129–153. Annals of mathematics studies, no. 34. Princeton University Press, Princeton, N. J., 1956. $4.00.

This paper studies distinguishability of sequential machines and distinguishability of internal states within a machine by experiments performed on the machines. It is shown that, for general machines, no finite experiment will suffice to identify one machine among the class of sequential machines (Theorem 2). By restricting the class of machines to those which are "strongly connected" and in "reduced form", it is then shown that there is some finite experiment with copies of a machine which will suffice to distinguish the machine from others in the class (Theorem 9). Several other theorems on distinguishability and on the length of experiments are proved.

C. Y. Lee (Chatham, N.J.).

★**Culbertson, James T. Some uneconomical robots.** Automata studies, pp. 99–116. Annals of mathematics studies, no. 34. Princeton University Press, Princeton, N. J., 1956. $4.00.

Using a number of each of three types of neurons, receptor, central and effector, and with proper interconnection of these neurons, the paper shows how a memoryless robot can be constructed. Then, by chopping time into discrete levels and increasing the number of receptor and central neurons, it is shown how a robot with complete memory (a record, at any time t, of all past receptor firings) can be constructed. A way of introducing probabilistic behavior without inserting unreliable elements is also indicated. *C. Y. Lee* (Chatham, N.J.).

Roster of automatic computers. Computers and Automation **5** (1956), no. 6, 86, 88, 90, 92, 94, 96.

★**Komamiya, Yasuo. Theory of the computing relay-networks.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 527–532. Science Council of Japan, Tokyo, 1952.

Let $A_1, A_2, \cdots, A_n$ represent n values chosen from {0, 1}; when used with the operator $+$, they will be numbers; when used with the operator $\vee$, they will receive a Boolean interpretation. Finally let m be the largest integer for which $2^m \leqq n$. Then the digits d_0 and d_1 in the dyadic expansion

$$A_1+A_2+\cdots+A_n=d_m2^m+d_{m-1}2^{m-1}+\cdots+d_12+d_0$$

are given by

$$d_0\equiv(A_n\equiv(A_{n-1}\equiv(\cdots\equiv(A_3\equiv(A_2\equiv A_1))\cdots))),\ \text{if}\quad n=2k+1,$$

$$d_0\equiv(\sim A_n\equiv(A_{n-1}\equiv(\cdots\equiv(A_3\equiv(A_2\equiv A_1))\cdots))),\ \text{if}\quad n=2k,$$

and

$$d_1 \equiv (B_{k1} \equiv (B_{k-11} \equiv (\cdots \equiv (B_{31} \equiv (B_{21} \equiv B_{11}))\cdots))), \text{ if } k \text{ is odd},$$

$$d_1 \equiv (\sim B_{k1} \equiv (B_{k-11} \equiv (\cdots \equiv (B_{31} \equiv (B_{21} \equiv B_{11}))\cdots))), \text{ if } k \text{ is even}.$$

where $B_{k1}, \cdots, B_{11}$ are determined by

$$A_1 + A_2 + A_3 = B_{11}2 + B_{10}$$
$$A_4 + A_5 + B_{10} = B_{21}2 + B_{20}$$
$$\vdots$$

ending with $A_{2k} + A_{2k+1} + B_{k-11} = B_{k1}2 + B_{k0}$ if n is odd, and $A_{2k} + B_{k-11} = B_{k1}2 + B_{k0}$ if n is even.

These expressions are used in the design of relay networks. The diagrams are not included in the paper.

S. Gorn (Philadelphia, Pa.).

Hunt, P. M. The electronic digital computer in aircraft structural analysis. The programming of the Argyris matrix formulation of structural theory for an electronic digital computer. II. The use of preset and programme parameters with the matrix interpretive scheme and their application to general purpose programmes for the force method of analysis. Aircraft Engrg. 28 (1956), 111–118.

Kautz, William H. Optimized data encoding for digital computers. Convention Record of the I. R. E. 2 (1954), part 4, 47–57.

Braun, Edward L. Design features of current digital differential analyzers. Convention Record of the I. R. E. 2 (1954), part 4, 87–97.

Frankel, Stanley. Useful applications of a magnetic-drum computer. Elec. Engrg. 75 (1956), 634–639.

ASTRONOMY

Masotti, Arnaldo, Su alculni probemi dinamici connettibili alla teoria degli ammassi stellari sferici. Mem. Soc. Astr. Ital. (N.S.) 27 (1956), 109–115.

Si considera il moto di un corpo celeste, assimilabile a un punto materiale, in presenza di un ammasso globulare, e si indicano due leggi per la densità dell'ammasso, aventi caratteri qualitativi comuni colla classica legge di Schuster-Plummer, alle quali corrispondono classi di orbite determinabili elementarmente. Si parla pure di un diagramma utile per la classificazione delle traiettorie in questo problema, e in altri analoghi.

Riassunto dell'autore.

Steffensen, J. F. On the differential equations of Hill in the theory of the motion of the Moon. II. Acta Math. 95 (1956), 25–37.

In part I [Acta Math. 93 (1955), 169–177; MR 17, 418] polar coordinates r, l were introduced to represent the orbit, and the corresponding differential equations were solved by power series in $x = \cos 2l$.

In the present paper the time t is used as the independent variable and the solution is given in powers of t or $\sin(t+\theta)$, where θ is an arbitrary constant and $|\sin(t+\theta)|$ is assumed to be sufficiently small. In both cases the coefficients of the power series are determined by recurrence formulas; and sufficient conditions for convergence of the series are given. Each case is illustrated by a numerical example.

E. Leimanis (Vancouver, B.C.).

Contopoulos, Georg. Beitrag zur Dynamik der Kugelsternhaufen. Z. Astrophys. 35 (1954), 67–73.

For a spherically symmetric density distribution that is continuous, positive, monotone decreasing, and vanishing at ∞ it is shown the orbits are of two types: 1) An open curve starting and ending at infinity with asymptotes meeting each other at an angle between 0 and π; 2) a curve that shuttles back and forth between a certain minimum and maximum distance from the origin with a constant angle (between $\frac{1}{2}\pi$ and π) between the radii vectors to a perihelion and a neighboring aphelion. In both cases the curves are concave with respect to the origin.

R. G. Langebartel (Urbana, Ill.).

Kurth, Rudolf. Stellar orbits in globular clusters. Astr. Nachr. 282 (1955), 241–246.

This paper establishes some elementary theorems on the motion of a test particle in a bounded, spherically symmetric, gravitating medium in which the density decreases outward. The most general orbit is an open rosette, whose points constitute a dense subset of an annular region centered on the origin. Orbits touching the boundary are necessarily rectilinear. [See also A. Wintner, The analytical foundations of celestial mechanics, Princeton, 1941; MR 3, 215; and the paper reviewed above.]

D. Layzer (Cambridge, Mass.).

Gurevič, L. È. Evolution of stellar systems. Voprosy Kosmog. 2 (1954), 150–260. (Russian)

The author's theory assumes initially a rotating flat distribution of matter. As stars start forming from this diffuse matter there begins a redistribution of the moment of momentum resulting in the formation of a spherical nucleus and a flattened outer system. Upon gradual contraction of the galaxy there is a greater divergence from the circular orbits in the galactic plane causing the galaxy to take on a spheroidal form. As these stars leave the diffuse matter in the equatorial plane new stars form in their place, thereby sustaining the dispersion to higher latitudes. In the process the axes of rotation of the protostars become widely scattered in direction and will not in general coincide with the magnetic axes of the fields imparted to the stars from the strong extensive magnetic field present in the diffuse nebula. There will be electric field vortices in the diffuse matter, and the collisions of these charged particles with atoms produce the complex elements.

R. G. Langebartel (Urbana, Ill.).

Davydov, B. I. On the dynamics of the galaxy. Astr. Ž. 32 (1955), 239–243. (Russian. English summary)

Inasmuch as the solution to the steady state Liouville equation for axial symmetry and symmetry about the equatorial plane depending on the two known integrals of motion appears inadequate to describe the observed motions in the neighborhood of the sun the author proposes a method of successive approximations based on the use of higher order moments to generate a more general solution. He writes out the equations satisfied by

the second order velocity moments but does not carry the analysis further. *R. G. Langebartel.*

Martin, E. L. Orbite anapsidali in sistemi binari di massa variabile. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 449–452 (1956).

Martin, E. L. Funzioni prive di estremanti in moti di sistemi binari di massa variabile. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **20** (1956), 56–59.

Michailovitsch, Dobrivoje. Eine Bemerkung über die Anwendungen der vektoriellen Elementen in der Theorie der sekulären Störungen. Bull. Soc. Math. Phys. Serbie **7** (1955), 229–232 (1956). (Serbo-Croatian. German summary)

Gottlieb, Ioan. Contributions à l'étude de la sphère matérielle de densité variable. I. L'étude du champ. Acad. R. P. Romîne. Fil. Iaşi. Stud. Cerc. Şti. **6** (1955), 209–216. (Romanian. Russian and French summaries)

L'auteur étudie dans ce travail, les sphères massives, dont les champs ont un minimum (ou extrémum) à l'intérieur de la sphère. Il établit ensuite la formule approximative des champs relativistes, dans l'hypothèse où la densité varierait suivant la loi de Roche.

Résumé français.

Krogdahl, Wasley S. Stellar pulsation as a limit-cycle phenomenon. Astrophys. J. **122** (1955), 43–51.

The author suggests a pulsating star to consist of a very large core which by expansion induces a shock wave in the thin surrounding envelope. The succeeding contraction is not to be of the shock wave type. For a mathematical description terms giving rise to instability and damping are arbitrarily inserted in the equation of motion which is then integrated numerically for several special cases. The differential equation is similar to the van der Pol equation. *R. G. Langebartel.*

Sobolev, V. V. Diffusion of radiation with redistribution of frequencies. I. Vestnik Leningrad. Univ. **10** (1955), no. 5, 85–100. (Russian)

In a gas so tenuous that the pressure is neglected the radiation diffusion is treated considering energy decay levels in the atom and the Doppler effect arising from the thermal motion of the atoms. The diffusion integral equation is set up and solved for three cases: 1) Radiation with no redistribution of frequencies; 2) radiation with complete redistribution; 3) radiation with the theoretically predicted amount of redistribution. Numerical integration is used to handle the last two cases. The radiation pressure and the contours of the spectral lines are determined for the three cases. Cases 2 and 3 give nearly the same results but these are materially different from those of case 1. *R. G. Langebartel* (Urbana, Ill.).

Zagar, Francesco. Modelli anisotropi nella cosmologia newtoniana. I. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 452–458.

The author sets up a hydrodynamical model of the universe, in which the velocity vector $\mathbf{v}=A\mathbf{r}$, where $\mathbf{r}$ is the position vector and A a symmetric matrix depending on the time t. Solutions are discussed in which the characteristic values of A are inversely proportional to t, and the position and velocity components increase with a power of t. It is concluded, with the aid of an inequality implied by the field equations, that for a mean density of matter of 10^{-29} gm/cm^3 the present epoch must be less than 10^{10} years. *H. P. Robertson* (Pasadena, Calif.).

Kwee, K. K.; and van Woerden, H. A method for computing accurately the epoch of minimum of an eclipsing variable. Bull. Astr. Inst. Netherlands **12** (1956), 327–330.

See also: Jordan, p. 1082; Koelbloed, p. 1094; Schrödinger, p. 1015.

RELATIVITY

Nohl, Walter. Kosmologische Lösungen eines homogenen Wirkungsprinzips. Comment. Math. Helv. **29** (1955), 338–350.

Solutions are sought for Scherrer's vector matter field equations [same Comment. **26** (1952), 184–202; **27** (1953), 157–164; MR **14**, 417; **15**, 170] under the assumption that the metric is of the form $ds^2=dx_0^2-L^2(x_0)d\sigma^2$, where $d\sigma$ is the metric of a three dimensional space of constant positive curvature and the matter field reduces to $(\varphi, 0, 0, 0)$. The author simplifies a very complicated set of equations obtaining three equations satisfied by L and φ. Eliminating φ gives a non-linear second order differential equation for L. This is solved most elegantly providing an interesting variety of cosmological behaviour corresponding to various possible choices of the constants of the theory. *A. J. Coleman* (Toronto, Ont.).

Reulos, René. Nonclassical transformation in special relativity. Phys. Rev. (2) **102** (1956), 535–536.

In a previous article [Ann. Inst. Fourier, Grenoble **5** (1953–54), 455–568, formula (1), p. 528; MR **17**, 930] we were led to consider a Lorentz transformation in the general case in which the velocity which generates the transformation is no longer parallel to one of the axes of the coordinates. It was noticed that the general form of this transformation, which is no longer linear with respect to the components of the velocity, could not be conveniently applied to certain problems of wave mechanics. We give here a general transformation, linear with respect to the components of the velocity, which is connected with Hamilton's quaternions and Dirac's matrices. Although the transformation is complex, it may, in certain problems, replace with advantage the Lorentz transformation.

In the realm of wave mechanics, our formalism enables us, by a simple transformation of the wave functions and the well-known Einstein relation $W=h\nu$, to obtain a very general system of equations which contains Proca's equations, de Broglie's equations of the photon, and the Maxwell-Lorentz equations. Our method leads us to consider the equations of electromagnetism as a natural generalization of the Cauchy-Riemann conditions and electromagnetic fields as generalizations of analytic functions, the variable being a quaternion.

Author's summary.

Duimio, Fiorenzo. Su una generalizzazione della dinamica relativistica della particella. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 18 (1955), 75–78.

This article presents a particular reformulation of particle dynamics, as the author himself says, which suggests various possibilities for further developments and applications. The main advantage ought to be the introduction of the mass as variable in a completely geometrized mechanical scheme. So the author starts from the action integral where the Lagrangian is of the form

$$L=\tfrac{1}{2}m_0\dot{x}_\mu\dot{x}^\mu+U(x_\mu,\dot{x}_\mu)$$

and neglects the condition $\dot{x}_\mu\dot{x}^\mu=-c t^2$. The motion of the particle is considered as taking place in a space of five dimensions, which is determined through the variables x_μ and $\tau_1=\tau/m_0$. He gives the Hamilton equation with respect to these coordinates. *T. P. Anđelić.*

Prosperi, Giovanni Maria. Sulle equazioni relativistiche del moto di una particella soggetta a forze derivanti da potenziale scalare. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 18 (1955), 69–74.

From a suitably generalized form of the action integral under the supplementary condition $\dot{x}_\nu\dot{x}^\nu=-c^2$ ($\nu=1, 2, 3, 4$) the author derives the wanted equations of motion in the form

$$(1)\qquad \frac{d}{d\tau}\left[\left(m_0+\frac{1}{c^2}\varphi\right)\dot{x}^\mu\right]=-\frac{\partial\varphi}{\partial x_\mu},$$

where φ denotes the potential function, while the other notations are usual. The four equations (1) are not all independent. The first three are sufficient to determine the motion of the particle, and the fourth can be interpreted as the equation of energy. It is then shown how the Hamilton form as well as the Hamilton-Jacobi form of the equations of motion can be derived. Lastly, the relations existing between the equations (1) and some problems of the relativistic hydrodynamics and wave mechanics are established. *T. P. Anđelić.*

Hlavatý, Václav. The elementary basic principles of the unified theory of relativity. C_4. General case. J. Rational Mech. Anal. 5 (1956), 419–472.

This is the final paper of the author's "C" series on the unified theory. The earlier ones appeared in the same J. 3 (1954), 103–146, 147–179, 645–689 [MR 15, 654; 16, 408]. With the asymmetric affine connection expressed in the form

$$\Gamma^\nu_{\lambda\mu}=\left\{{\nu \atop \lambda\mu}\right\}+S_{\lambda\mu}{}^\nu+U^\nu{}_{\lambda\mu},$$

the earlier papers have dealt with cases in which $U^\nu{}_{\lambda\mu}$ has special forms. In the present paper the author takes the case of a general $U^\nu{}_{\lambda\mu}$, and also replaces the Einsteinian condition

$$R_{\mu\lambda}=\partial_{[\mu}x_{\lambda]}$$

(see paper C) by the more general condition

$$R_{\mu\lambda}=\partial_{[\mu}x_{\lambda]}-\phi h_{\lambda\mu},$$

ϕ being an unspecified scalar. The paper follows the same general lines as the earlier ones, distinct cases being examined separately. Both electromagnetic and gravitational field equations are obtained, the two fields being interrelated. The electromagnetic field is always present, and may be regarded as a primary field which, in general, creates a gravitational field. Nevertheless, there exist electromagnetic fields that do not create gravitation. The "unified law of inertia" states that a particle moving freely in a unified field describes an autoparallel line of the connection $\Gamma^\nu_{\lambda\mu}$. There exist coordinate-systems for which this law reduces in the first approximation to the classical Newtonian law of gravitation and in the second to the classical-relativity second Newtonian law for the electromagnetic force vector.

The paper ends with a short appendix devoted to the elementary geometry of the Michelson-Morley experiment as suggested by the unified theory and on the assumption that the velocity of light is constant. The latter assumption does not exclude, at least theoretically, the possibility that D. C. Miller's results [P. G. Bergmann, Introduction to the theory of relativity, Prentice-Hall, New York, 1942, p. 133; MR 4, 55] are consistent with those of Michelson and Morley, a conclusion that the author describes as astonishing. *H. S. Ruse* (Leeds).

Pham Mau Quan. Sur une théorie relativiste des fluides thermodynamiques. Ann. Mat. Pura Appl. (4) 38 (1955), 121–204.

In this paper the author develops in considerable detail ideas presented previously in three short notes [C. R. Acad. Sci. Paris 236 (1953), 2299–2301; 237 (1953), 22–24; 238 (1954), 324–325; MR 14, 1134, 1135; 15, 752]. He deals for the most part with a perfect fluid for which the following equations hold (Greek suffixes=0, 1, 2, 3):

$$T^{\alpha\beta}=(\varrho+p)u^\alpha u^\beta-pg^{\alpha\beta}-(u^\alpha q^\beta+u^\beta q^\alpha),$$
$$q_\alpha=-\varkappa\partial_\varrho\theta(g_\alpha{}^\varrho-u^\varrho u_\alpha),$$
$$\nabla_\alpha q^\alpha=c\varrho u^\alpha\partial_\alpha\theta-\varrho^{-1}lu^\alpha\partial_\alpha\varrho,$$
$$u^\alpha u_\alpha=1,\ \varrho=\phi(p,\theta).$$

Here ϱ=proper density, p=pressure, θ=temperature, u^α=4-velocity, q^α=4-vector of heat flow, $\varkappa$=thermal conductivity, c=specific heat at constant volume, l=heat of dilatation, these last three being regarded as known functions of ϱ and θ. The above equations are combined with the usual field equations of general relativity ($S_{\alpha\beta}=\chi T_{\alpha\beta}$). [We have then a system of 17 equations for the 21 unknowns ($\varrho, p, \theta, u^\alpha, q^\alpha, g_{\alpha\beta}$), so that the problem of the fluid is well stated mathematically, since there are 4 additional coordinate conditions to be imposed on $g_{\alpha\beta}$.] The author examines the Cauchy problem for this system of equations, and shows that the required Cauchy data on a 3-space S are $g_{\alpha\beta}$, $\partial_\lambda g_{\alpha\beta}$, θ, $\partial_\lambda\theta$, unless S is a characteristic 3-space or a shock wave; these special cases are discussed in detail. The paper has five chapters, of which I, II and IV deal with the theory outlined above and V with permanent movements and stationary models of thermodynamic universes. Chapter III, devoted to the general principles of relativistic thermodynamics, follows a different and less deductive pattern. Here the discussion proceeds first in Newtonian terms, introducing density of matter (μ) and specific internal energy (ε). Next, the equations of conservation $\nabla_\beta T^{\alpha\beta}=0$ in flat space-time are used to obtain relativistic equations of motion, and a comparison suggests to the author the relationship $\varrho=\mu(1+\varepsilon)$, so that the proper density ϱ is to be interpreted as the density of matter and energy combined. There is also some discussion of proper density of entropy (σ), but the reviewer is unable to see the significance of μ, ε and σ relative to the general theory of the other chapters, which appears to be complete without them. *J. L. Synge.*

Pham Mau Quan. Etude électromagnétique et thermodynamique d'un fluide relativiste chargé. J. Rational Mech. Anal. **5** (1956), 473–538.

This paper, which follows the general methods of A. Lichnerowicz [Théories relativistes de la gravitation et de l'électromagnétisme, Masson, Paris, 1955; MR **17**, 199], expands ideas presented in previous notes [C. R. Acad. Sci. Paris **240** (1955), 598–600, 733–735; MR **16**, 872; see also the paper reviewed below], and rests to some extent on the paper reviewed above. In addition to the hydrodynamical quantities involved in that paper, we now have two skew-symmetric electromagnetic tensors $G_{\alpha\beta}$, $H_{\alpha\beta}$, a 4-current J^α, and four scalars λ, μ, σ, δ, corresponding to dielectric constant, permeability, electric conductivity. and proper density of electric charge. [The functional status of these scalars is not made clear; λ, μ, and σ appear to be regarded as known, whereas δ is an unknown.] In addition to equations quoted in the preceding review, we have the following:

$$\nabla_\alpha G_\beta{}^\alpha = J_\beta,\ \partial_\gamma H_{\alpha\beta}+\partial_\alpha H_{\beta\gamma}+\partial_\beta H_{\gamma\alpha}=0,$$

$$G_{\alpha\beta}u^\alpha=\lambda H_{\alpha\beta}u^\alpha,\ \mu G_{\alpha\beta}{}^*u^\alpha=H_{\alpha\beta}{}^*u^\alpha,$$

$$J_\beta=\delta u_\beta+\sigma u^\varrho H_{\varrho\beta}.$$

The system of equations is completed by the usual field equations $S_{\alpha\beta}=\chi T_{\alpha\beta}$, and all that remains is to choose the energy tensor $T_{\alpha\beta}$. The author defines

$$T_{\alpha\beta}=(\varrho+p)u_\alpha u_\beta-pg_{\alpha\beta}-(u_\alpha q_\beta+u_\beta q_\alpha)+\tau_{\alpha\beta}-(1-\lambda\mu)\tau_{\alpha\varrho}u^\varrho u_\beta,$$

where

$$\tau_{\alpha\beta}=\tfrac{1}{4}g_{\alpha\beta}G_{\varrho\sigma}H^{\varrho\sigma}-G_{\varrho\alpha}H_\beta{}^\varrho,$$

thus combining a mechanical part with the Minkowskian $\tau_{\alpha\beta}$ and adding an ad hoc interaction to make $T_{\alpha\beta}$ symmetric in spite of the lack of symmetry of $\tau_{\alpha\beta}$. He examines the Cauchy problem and finds the required Cauchy data on a 3-space S to be $g_{\alpha\beta}$, $\partial_\lambda g_{\alpha\beta}$, θ, $\partial_\lambda\theta$, $H_{\alpha\beta}$, unless S is exceptional, i.e. a characteristic gravitational 3-space, a hydrodynamical shock wave, or an electromagnetic characteristic 3-space. Among other matters considered is the problem of a frontier between two different fluids.

J. L. Synge (Dublin).

Pham Mau Quan. Sur les équations de l'électromagnétisme dans la matière. C. R. Acad. Sci. Paris **242** (1956), 465–467.

In the space-time of general relativity with metric $g_{\alpha\beta}dx^\alpha dx^\beta$, an electromagnetic field is described by two skew-symmetric tensors $G_{\alpha\beta}$, $H_{\alpha\beta}$, and for an electromagnetic fluid with 4-velocity u^α [cf. Pham Mau Quan, same C. R. **240** (1955), 598–600, 733–735; MR **16**, 872] we have the equations

$$u^\alpha G_{\alpha\beta}=\lambda u^\alpha H_{\alpha\beta},\ \mu u^\alpha G_{\alpha\beta}{}^*=u^\alpha H_{\alpha\beta}{}^*,$$

where * means dual or adjoint, and λ, μ are scalars. Maxwell's equations read

$$\nabla_\alpha H^{*\alpha\beta}=0,\ \nabla_\alpha G^{\alpha\beta}=J^\beta,\ J^\beta=\delta u^\beta+\sigma u_\alpha H^{\alpha\beta},$$

where δ, σ are scalars. A study of the structure of the field equations shows that the characteristic 3-spaces $f(x^\alpha)=0$ satisfy

$$\gamma^{\alpha\beta}\partial_\alpha f\partial_\beta f=0,\ \gamma^{\alpha\beta}=g^{\alpha\beta}-(1-\lambda\mu)u^\alpha u^\beta.$$

In terms of this tensor the two basic tensors are related by

$$G^{\alpha\beta}=\mu^{-1}H_{\varrho\sigma}\gamma^{\varrho\alpha}\gamma^{\sigma\beta}.$$

Defining $\gamma_{\alpha\beta}$ by $\gamma^{\alpha\sigma}\gamma_{\sigma\beta}=\delta_\beta{}^\alpha$, so that

$$\gamma_{\alpha\beta}=g_{\alpha\beta}-(1-(\lambda\mu)^{-1})u_\alpha u_\beta,$$

the author concludes that electromagnetic rays are null geodesics for the metric $\gamma_{\alpha\beta}dx^\alpha dx^\beta$. [This provides the field background for the geometrical optics of N. L. Balazs, J. Opt. Soc. Amer. **45** (1955), 63–64; MR **16**, 872.]

J. L. Synge (Dublin).

Pham Mau Quan. Projections des géodésiques de longueur nulle et rayons électromagnétiques dans un milieu en mouvement permanent. C. R. Acad. Sci. Paris **242** (1956), 875–878.

Under stationary conditions in space-time, the null geodesics for the metric $\gamma_{\alpha\beta}dx^\alpha dx^\beta$ [see the preceding review] yield a variational principle in the three spatial variables (Fermat's principle): it reads

$$\delta\int[\varepsilon(\gamma_{00})^{-\frac{1}{2}}(-\gamma_{ij}\dot{x}^i\dot{x}^j)^{\frac{1}{2}}-\gamma_{00}{}^{-1}\gamma_{0i}\dot{x}^i]du=0,$$

$$\hat{\gamma}_{ij}=\gamma_{ij}-\gamma_{0i}\gamma_{0j}\gamma_{00}{}^{-1}\ (i,j=1,2,3),\quad \varepsilon=\text{sign }\gamma_{0\alpha}\dot{x}^\alpha.$$

J. L. Synge (Dublin).

Géhéniau, J. Les invariants de courbure des espaces riemanniens de la relativité. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 252–255.

The author gives a number of formulae that lead to simplified expressions for the curvature-invariants of Riemannian 4-space obtained by him and Debever in a previous paper [same Bull. (5) **42** (1956), 114–123; MR **17**, 1016]. *H. S. Ruse* (Leeds).

Vaidya, P. C. The general relativity field of a radiating star. Bull. Calcutta Math. Soc. **47** (1955), 77–80.

The author derives a form of line element in general relativity theory which can be applied to the field of a radiating star. *M. Wyman* (Edmonton, Alta.).

Ikeda, Mineo. On boundary conditions in the nonsymmetric unified field theory. Progr. Theoret. Phys. **15** (1956), 1–11.

The author considers the long standing problem of the formulation of covariant boundary conditions in relativistic field theories. By first formulating a covariant concept of special infinity a meaning is attached to the approach of one tensor to another at spacial infinity. A comparison is then made of the implications of the new concept and the usual non-covariant one that has been generally used. *M. Wyman* (Edmonton, Alta.).

McCrea, W. H.; and Mikhail, F. I. Vector-tetrads and the creation of matter. Proc. Roy. Soc. London. Ser. A. **235** (1956), 11–22.

Starting with a four dimensional space of events the authors associate with each event four linearly independent contravariant vectors $\lambda_m{}^\mu$, where m indicates the vector and μ the components of the vector. With this as basis a symmetric metric tensor is defined and possible field equations are considered. These equations are more general than the field equations of classical relativity in that a term appears which allows for the possibility of the creation of matter. In addition several particular examples are considered in detail. *M. Wyman.*

Caldirola, P. A new model of classical electron. Nuovo Cimento (10) **3** (1956), supplemento, 297–343.

The equations of the two-body problem in general

relativity are derived, to order K^4 in an expansion of the gravitational constant K. Unlike the Einstein-Infeld-Hoffmann [Ann. of Math. (2) 39 (1938), 65–100] method, an approach from conventional linear field theory is employed on the linearized Einstein equations. The treatment is given in quantum language, although to this order the classical and quantum results are quite similar. The Einstein-Infeld-Hoffmann equations are found to hold for the motion of wave-packets, so that to this order the linearized and usual theory coincide. It is pointed out that the true difficulties of current field theory would arise in the next order of the linear approximation. *S. Deser* (Copenhagen).

Bechert, Karl. Nichtlineare Feldtheorie. Z. Naturf. **11a** (1956), 177–182.

The author gives a somewhat detailed discussion of the properties of a gravitation theory where the matter tensor is written as a sum of an electromagnetic part of the conventional form and another term of the form $UV_\mu V_\nu$. The quantity U is a scalar and V_μ a four vector that fulfills the identity $V_\mu V^\mu=1$. The modification of the Schwarzschild solution due to the presence of the electromagnetic term is discussed. The author seems to be unaware of the fact that these things are also discussed in several well-known text books, e.g. in the book by Weyl [Raum, Zeit, Materie, 4th ed., Springer, Berlin, 1921]. Eq. (5.2) in the paper is e.g. also given in chapter IV, § 32, p. 236 of Weyl's book. *G. Källén* (Copenhagen).

Majorana, Quirino. Su di una nuova teoria della gravitazione. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **19** (1955), 95–101.

Kustaanheimo, Paul. Some remarks on the general relativity theory of Birkhoff. Soc. Sci. Fenn. Comment. Phys.-Math. **17** (1955), no. 11, 15 pp.

The author discusses the set of axioms of Barajas and Graef Fernandez for Birkhoff's theory [see Graef Fernández, Symposium sobre algunos problemas matemáticos que se están estudiano en Latino América, 1951, Centro de Cooperación Cientifica de la Unesco para América Latina, Montevideo, Uruguay, 1952, pp. 121–137; MR **14**, 807]. He finds that this set is satisfied by a model depending upon three parameters, of which one must be so chosen that the model gives Newtonian gravitation as a first approximation. Of the others, one may be chosen so that the predicted result of one of the three crucial tests has an arbitrary value. The author gives a set of axioms equivalent to those of Barajas and Graef Fernández, and remarks that a slight modification of these leads to a set admitting a 6-parameter model. Of the new parameters, two may be so chosen that the predicted results of the other two crucial tests are also arbitrary.

H. S. Ruse.

See also: Carini, p. 1159; Corinaldesi, p. 1165.

MECHANICS

Rašković, Danilo. On some characteristics of the frequency equation of holonomic conservative system with static constraints. Zb. Mašin. Fak. **1953-54**, 65–77. (Serbo-Croatian. English and French summaries)

In this exercise the author considers systems whose double energy has the form $a_{ij}\dot q_i \dot q_j + c_{ij} q_i q_j$ $(i, j=1, \cdots, n)$, a_{ij} being the elements of the unit matrix, $c_{ij}=0$ for $|i-j|>1$, $c_{ij}=-c$ for $|i-j|=1$, $c_{ij}=2c$ for $i=j\neq 1, n$. For the special cases,

$$c_{11}=c_{nn}=c,\quad c_{11}=c_{nn}=2c,\quad c_{11}=2c_{nn}=c,$$

(referred to, it would seem, by means of the supserscripts s, u, k, respectively) the frequencies are given by $4p\sin^2(m\pi/2)$ where, respectively, $m=i/n$, $i/(n+1)$, $(2i-1)/(2n+1)$. The frequency equation is derived from recurrence relations. The summaries do not agree with the paper.

A. W. Wundheiler (Chicago, Ill.).

Rosenauer, N. Anwendung von komplexen Veränderlichen zur Synthese einer Kurbelschwinge mit vorgeschriebenen Grenzen der Abtriebs-Winkel-Geschwindigkeit. Ing.-Arch. **24** (1956), 43–46.

Vereinfachung einer vom Verfasser herrührenden Lösung [Mabau.-Getriebetechnik **12**, (1944) 25–27]. Die Glieder des Gelenkvierecks werden durch komplexe Zahlen dargestellt. Die Summe ist Null; durch Differentiation nach der Zeit entstehen zwei Gleichungen für die Winkelgeschwindigkeiten und die Beschleunigungen. Es wird daraus bei gleichförmigem Antrieb und für extreme Werte des Abtriebes eine kubische Gleichung für die Geschwindigkeit des Koppelgliedes abgeleitet. Anwendung auf ein Zahlenbeispiel.

O. Bottema (Delft).

Rosenauer, N. Synthesis of four-bar linkages with prescribed reduction ratio limits: application of complex variables. Austral. J. Appl. Sci. **7** (1956), 1–9.

This is nearly identical with the paper reviewed above with the same numerical example. *O. Bottema.*

Freudenstein, Ferdinand. On the maximum and minimum velocities and the accelerations in four-link mechanisms. Trans. A.S.M.E. **78** (1956), 779–787.

Given a four-bar linkage $ABCD$; DA is the fixed, AB the driving, BC the connecting, CD the follower link, the pole P is the intersection of AB and CD, O that of BC and AD, PO is the collineation axis. Author is interested in the ratio of the angular velocity of AB and CD and proves by geometrical reasoning that at an extreme value OP is perpendicular to BC. Construction of linkages with prescribed extreme value. Extension to acceleration analysis of the linkage. Application to slider-crank mechanisms. Appendix contains a proof of the main theorem by means of Bobillier's. Discussion added (pp. 786–87). *O. Bottema* (Delft).

Veiga de Oliveira, F. On representation of rotations by means of matrices. Univ. Lisboa. Revista Fac. Ci. A. (2) **5** (1955–1956), 119–134. (Portuguese)

The author formulates the elementary kinematical theory of a solid rotating about a fixed point in terms of matrix algebra. No essentialy new results are given.

L. A. MacColl (New York, N.Y.).

Gol'din, E. M. On motion of a material point inside a rapidly rotating cone. Izv. Akad. Nauk SSSR. Otd. Tehn. Nauk **1955**, no. 6, 72–90. (Russian)

The problem is suggested by technological separation processes. A circular cone of apex angle 2α is rotating at

a constant angular velocity ω about its axis, while a light particle M is moving on its inner surface. Let m be the mass of M, and ϱ, ϕ, α its polar coordinates ($\varrho=0$ is the apex, and $\alpha=0$ the axis of the cone). If the resistance is viscous, and its coefficient is $m\gamma$, then putting

$$x=1+\dot{\phi}/\omega,\ \dot{\varrho}/(\varrho\omega \sin \alpha),\ \tau=t\omega \sin \alpha,\ k=\gamma/(2\omega \sin \alpha),$$

the equation

$$dz/d\tau=iz^2-2kz+2k$$

is obtained, where $z=x+iy$. Its exact solution is replaced by two approximations, for large and small γ, and the behavior of the particle is discussed.

In the case of dry friction, the author puts $b=f \cot \alpha$ (f-coefficient of friction), and uses the same variables, x, y, τ. There are three singular points in the plane (x, y): $(0, 0)$, $(1, 0)$, and $[2/b-1, (2/b-1)(b-1)^{\frac{1}{2}}]$, which represent motions at constant x and y. These motions are, respectively; absolute rest, stable relative rest (jamming), and unstable relative rest. For $b<1$ the absolute velocity tends towards zero, for $b\geqq 1$ either the absolute or the relative velocity tends toward zero. There is much detail in the paper, including the distribution of the hodographs in the plane (x, y). *A. W. Wundheiler.*

Cypkin, Ya. Z. On a connection between the describing function of a nonlinear element and its characteristic. Avtomat. i Telemeh. **17** (1956), 343–346; appendix to no. 4, 4. (Russian. English summary)

A non-linear element of a non-linear control system is considered to be represented by $x_2=F(x_1)$, where x_1 and x_2 are the element input and output respectively, and the function $F(x)$ describes the characteristic behavior of the element. For $F(x)$ odd and input $x_1=A \cos wt$, harmonic linearization of $F(x_1)$ yields

$$(*)\qquad F(x_1)=S(A)x_1$$

$$(**)\qquad S(A)=\frac{2}{\pi A}\int_0^\pi F(A \cos \psi)\cos \psi\, d\psi.$$

Attention is centered on deriving approximate formulas to replace (**). In particular, calculations are made comparing (**) with

$$S(A)=\frac{2}{3A}\Big[F(A)+F\Big(\frac{A}{2}\Big)\Big],$$

obtained using an approximate integration formula of V. A. Stekloff. *J. F. Heyda* (Indianapolis, Ind.).

Litwiniszyn, J. On dislocations of granular media. Zastos. Mat. **2** (1956), 380–389. (Polish. Russian and English summaries)

Colombo, Giuseppe. Riduzione alle quadrature di un notevole problema di stereodinamica. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) **18** (1955), 168–172.

The author considers the motion under gravity of a solid C of gyroscopic structure a part of which is a sphere S with its center on the axis of C; S is placed on a rough horizontal plane on which it is able to roll without sliding. For this nonholonomic system of a top with spherical base we have the energy-integral and a second first integral given already by Jellet, [Theory of friction, Dublin-London, 1905, pp. 189–192]. The author adds a third first integral and so is able to reduce the problem to quadratures. *O. Bottema* (Delft).

Fluid Mechanics, Acoustics

Brillouin, L. Les pressions de radiation et leur aspect tensoriel. J. Phys. Radium (8) **17** (1956), 379–383.

This paper and the four following are contributions from a symposium on the definition, nature, and measurement of radiation pressure in acoustics, held in May of 1955 at Marseille. It is appropriate that in the first paper Professor Brillouin discusses what he himself was first to point out years ago [see Les tenseurs en mécanique et en élasticité, Masson, Paris, 1938, pp. 288–302; MR **8**, 97], that radiation pressure has actually the character of a stress system and requires a tensor for its specification. In the case of an unperturbed plane wave, one can compute the tensor of the average stresses either from the equations of wave propagation while retaining non-linear terms or from the Boltzmann-Ehrenfest formula. The latter is particularly useful when the medium is dispersive. From this tensor the resultant force on an obstacle in the radiation field can be obtained. Several examples are discussed, including the author's application to the calculation of the scattering cross-section of a sphere [J. Appl. Phys. **20** (1949), 1110–1125; MR **11**, 562]. *R. N. Goss* (San Diego, Calif.).

Mawardi, Osman K. Sur la pression de radiation en acoustique. J. Phys. Radium (8) **17** (1956), 384–390.

The author shows how the tensorial character of radiation pressure follows from the equations of motion and continuity of a perfect non-viscous fluid. The radiation tensor is the sum of two parts: the one being the average flux of momentum, the other arising from the nonlinearity of the propagation medium. The presence of the latter makes the exact calculation of the tensor components contingent upon the solution of a non-linear wave equation. For plane waves and a particular equation of state the tensor is computed approximately by means of a transformation between the two functions expressing the same property in Lagrangian and Eulerian coordinates. Under the same approximation a formula for the force exerted on an obstacle in a sound field is developed. The scattering cross-section is discussed, and numerical results for a sphere are given. Finally it is shown how the viscosity of the medium can be taken into account. *R. N. Goss* (San Diego, Calif.).

Post, E. J. La pression de radiation acoustique. J. Phys. Radium (8) **17** (1956), 391–394.

In studying the difference between the two parts of the radiation tensor [see the preceding review] one is led to the concept of a membrane permeable to the medium but impermeable to the radiation — the analog of a mirror in optics. The author employs the Boltzmann-Ehrenfest principle to derive an expression for the radiation tensor on the hypothesis of the existence of such a semi-permeable membrane. He then justifies the validity of the concept by showing that first-order relativistic transformations of the transport energy and transport momentum are independent of the medium. The notion of a phonon is likewise shown to be equivalent to that of a semi-permeable membrane. *R. N. Goss* (San Diego, Calif.).

Lucas, R. Les tensions de radiation en acoustique. J. Phys. Radium (8) **17** (1956), 395–399.

The method used to study the stresses in a strained elastic body is applied to a non-viscous fluid disturbed by sound waves. The connection between the results and the

radiation tensor obtained from the Boltzmann-Ehrenfest principle is pointed out. The ratio of average sound pressure to acoustic energy density is found for plane and for spherical waves. The problems of molecular pressure of a perfect gas and osmosis through a semi-permeable membrane are treated from the viewpoint of acoustic radiation in order to emphasize the broad applicability of the concept of radiation pressure. *R. N. Goss.*

Johansen, A. Force agissant sur une sphère suspendue dans un champ sonore. J. Phys. Radium (8) **17** (1956), 400.

The tensor from which one obtains the flux of momentum in a sound field is calculated. It is used to find the resultant force acting on a rigid sphere suspended in an infinite plane-wave field. Numerical results for various ratios of radius to wavelength are tabulated.

R. N. Goss (San Diego, Calif.).

Cady, Walter G. Theory of the plane wave acoustic filter with periodic structure. Akad. Wiss. Mainz. Abh. Math.-Nat. Kl. **1955**, 219–234.

A plane sound wave enters normally a medium consisting of $2n$ homogeneous strata separated by parallel planes, each alternate stratum having the same thickness and acoustic parameters. The structure is consequently periodic in sections of two strata each. Let l be the width of a section. What amounts to the group property of the bilinear transformation is used to obtain a formula expressing the specific acoustic impedance at the plane $x=nl$ in terms of that at the plane $x=0$ with an iteration factor $\varkappa^n$, where $\varkappa$ depends on the constants of the two strata comprising a section. From this formula the coefficient of reflection at $x=nl$ is obtained. The special case of a filter in which the width of each stratum is the same fraction of a wavelength is discussed. *R. N. Goss.*

Heyda, James F. An elementary derivation of the formula for the windage jump of a spinning shell. J. Franklin Inst. **261** (1956), 615–619.

Poincelot, Paul. Réflexion des signaux radioélectriques sur l'ionosphère. Ann. Télécommun. **11** (1956), 70–80.

The author studies the propagation of plane electromagnetic waves in a stratified medium under two hypotheses regarding the dependence of the index of refraction upon height: (i) $n^2=1-\alpha z/\omega^2$; (ii) $n^2=1-(a^2/\omega^2)(1-e^{-\alpha z})$, where n is the index, z ($\geqq 0$) is the height, and a, α, ω are constants. For $z<0$ the medium is homogeneous with index unity. In each case a series of transformations on the resulting time-independent wave equation in the electric field intensity leads to a form whose solution is a linear combination of cylinder functions. Boundary conditions are applied, and the coefficient of reflection from the plane $z=0$ is calculated. The case of oblique incidence is discussed. The theory of group velocity is used to extend the results to pulse trains, and a numerical example is given. This paper brings together and amplifies results which have been previously reported by the author in a series of fragmentary notes. *R. N. Goss.*

Robinson, Abraham. On the motion of small particles in a potential field of flow. Comm. Pure Appl. Math. **9** (1956), 69–84.

Bei der Untersuchung der Bewegung fester Teilchen in einer gegebenen Strömung interessierte man sich bisher meistens für die Bahnen der einzelnen Partikel. Im Gegensatz hierzu betrachtet der Verf. dasjenige (virtuelle) Strömungsfeld, das einer kontinuierlichen Verteilung kleiner Teilchen von gegebener Größe in einer Potentialströmung entspricht. Die Bewegungsgleichung beruht auf der Annahme, daß die Grundströmung auf ein Teilchen einen Widerstand ausübt, der der Geschwindigkeitsdifferenz zwischen Grundströmung und Teilchen gemäß dem Stokesschen Gesetz proportional ist. Im Anschluß hieran werden einige allgemeine Eigenschaften der Partikelströmung, insbesondere ein Zirkulationssatz, abgeleitet und die Strömung hinter einem schlanken Hindernis (nach Linearisierung) untersucht. Im Falle einer ebenen Strömung läßt sich für die Masse derjenigen Teilchen, die in der Zeiteinheit einen gegebenen Teil der Kontur des (schlanken) Hindernisses überstreichen, eine geschlossene Formel angeben. *K. Maruhn* (Dresden).

Čirkin, M. P. On the computation of the slip of a viscous fluid along rigid walls. Akad. Nauk Uzbek. SSR. Trudy Inst. Mat. Meh. **10** (1953), no. 2, 160–178. (Russian)

The author's work is predominantly of an expository character. The author does not agree with the general notion of no slipping at the rigid walls of tubes, etc. [cf. Lamb, Hydrodynamics, 6th ed., Cambridge, 1932, p. 576]. Consequently, he tries to develop a theory along the lines of N. P. Petrov [cf. N. E. Kočin, I. A. Kibel' and N. V. Roze, Theoretical hydrodynamics, Gostehizdat, Moscow, 1948, p. 419; MR **17**, 307] to this effect. The author exemplifies his methods by treating — in great detail — several different types of motion of viscous incompressible fluids, viz., a) the Poiseuille type of flow through tubes of circular cross-section, b) the flow between two rotating concentric cylinders, c) the one-dimensional flow between two plane (rigid) parallel walls, d) the slow motion of a sphere, e) the slow rotation of a sphere.

It should be pointed out in this connection that none of the solutions given by the author are new; also, his arguments related to slipping are not convincing.

K. Bhagwandin (Oslo).

Mhitaryan, A. M. On filtration of water through an earthen dam with shield and spillway on a permeable base with an inclined waterproof foundation. Ukrain. Mat. Ž. **6** (1954), 448–456. (Russian)

Birkhoff, Garrett; and Kotik, Jack. Theory of the wave resistance of ships. II. The calculation of Michell's integral. Trans. Soc. Naval Arch. Marine Engrs. **62** (1954), 372–385.

Michell's theory gives the wave resistance of a thin ship in smooth water as a quintuple integral, the integrand being an oscillatory function. The accurate numerical evaluation of the integral is difficult. The authors review existing methods and propose two new ones, based on transformations of the integral, which they have found advantageous. Some instructive comparisons are given.

F. Ursell (Cambridge, England).

Birkhoff, Garrett; and Kotik, Jack. Some transformations of Michell's integral. Publ. Nat. Tech. Univ. Athens no. **10** (1954), 26 pp. (Greek summary)

Various forms of Michell's integral [see the preceding review] are proved rigorously to be equal to each other.

F. Ursell (Cambridge, England).

Boreli, Mladen. Sur une solution rigoureuse d'un problème d'écoulement plan en milieu poreux avec barrage souterrain. C. R. Acad. Sci. Paris **239** (1954), 1020–1021.

The analysis of this problem is reproduced in the paper reviewed below. *K. Bhagwandin* (Oslo).

Boreli, Mladen M. Contribution à l'étude des milieux poreux. Publ. Sci. Tech. Ministère de l'Air, Paris, no. **305** (1955), vii+131 pp. 1700 francs.

The present work constitutes the author's doctoral thesis (Grenoble). A good deal of the work is of an expository character. The author deals primarily with the rectilinear flow of heavy liquids in homogeneous and isotropic porous media. The study is based on Darcy's law, and its operational region in infiltration theory is established. This discussion constitutes the first part of the author's work. The second part deals with plane motion. Elementary conformal mapping theory, the hodograph method (the author does not seem to have heard of S. N. Čaplygin and his school), and Levi-Civita's transformation are sketched. It should be pointed out in this connection that M. A. Lavrent'ev has also dealt with similar problems [Mat. Sb. N.S. **4**(46) (1938), 391–458; Akad. Nauk Ukrain. RSR Zb. Prac' Inst. Mat. **1946**, no. 8, 13–69 (1947); C. R. (Dokl.) Acad. Sci. URSS (N.S.) **41** (1943), 275–277, MR **14**, 102; **6**, 191; **15**, 906]. By means of these methods, the author studies the fundamental problem of the free-surface of different types of ground-water flow, etc. Much attention is paid to the problem of singularities [cf. the paper reviewed below]. The flow towards a drain, placed on an impermeable foundation is considered, and the related free- surface problem is discussed. The results are plotted in graphs. The author also draws an analogy between Darcy's law $\mathbf{v}=-K\,\overrightarrow{\mathrm{grad}}\,h$ and the formula $\mathbf{i}=-c\,\overrightarrow{\mathrm{grad}}\,E$ of electro-statics. The third part (the author's major contribution) deals with the application of relaxational methods to flow into porous media [cf. McNown, Hsu and Yih, Proc. Amer. Soc. Civil Engrs **79** (1953), no. 223]. Finally the author studies the problem of gravity wells by means of relaxational methods. The author's results are displayed in numerous curves (74 figs.) which seem to be reasonable. *K. Bhagwandin.*

Kravtchenko, J.; Sauvage de Saint-Marc, G.; et Boreli, M. Sur les singularités des écoulements plans et permanents des nappes souterraines pesantes. Houille Blanche **10** (1955), 47–62.

The author's work is based on the assumption that the conformal pattern in the neighbourhood of the singularity is known. These singularities are then studied in detail by means of classical methods. In the case of three singularities, the problem can be dealt with in terms of hypergeometric functions. The authors' work seems to be an elementarization of some of Polubarinova-Kočina's techniques [cf. Theory of motion of ground water, Gostehizdat, Moscow, 1952; MR **15**, 71]. Proofs and error estimates are not given. *K. Bhagwandin* (Oslo).

Kravtchenko, J.; Sauvage de Saint-Marc, G.; et Boreli, M. Etude d'une singularité dans les écoulements plans des liquides pesants en milieux poreux. Houille Blanche **10** (1955), 533–542.

This paper is a continuation of the work reviewed above. In the present paper the authors apply their methods to the case of a point common to a free surface and an impermeable wall. The analysis is still theoretical. *K. Bhagwandin* (Oslo).

Nasyrov, R. M. Determination of the shape of a biplane for a given velocity distribution on the surface of the profiles constituting it. Kazan. Gos. Univ. Uč. Zap. **113**, no. 10 (1953), 31–41. (Russian)

Suppose that in the $z=x+iy$ plane the speed of flow is given as a function of arc length on each profile, the values of the complex potential w are known at the forward stagnation points, and the arc lengths at the trailing edges are given. The author conformally maps the desired incompressible flow about the unknown profiles as follows onto a flow about two known profiles in a u-plane. Define $w(u)$ by $dw/du=f(u)$, where f is a certain elementary function of u and eight parameters, and $\mathrm{Re}\,uf(u)=0$ on two arcs q_1p_1 and q_2p_2 of $|u|=1$. Values of the parameters such that q_1p_1 and q_2p_2 will become the images of the unknown profiles can be defined implicitly by the following conditions. Choose $\mathrm{Im}\,q_1=0$; demand dw/du be regular at $u=0$; assign the known circulations to q_1p_1 and q_2p_2; and impose the known values of w at images of stagnation points and trailing edges. If the resulting equations actually have physically acceptable solutions, a question not considered by the author, then the values of $|dz/du|^+=|dw/du|^+/|dw/dz|^+$ and of $|dz/du|^-$ can be found on q_1p_1 and q_2p_2. This finally leads to the problem to determine $\chi(u)=\log dz/du$ bounded at infinity and with prescribed boundary values $\mathrm{Re}\,\chi^+$ and $\mathrm{Re}\,\chi^-$ on q_1p_1 and q_2p_2. $\chi(u)$ can be found by quadratures provided two integrals involving the boundary values vanish. If the velocity $v_0e^{i\theta}$ at $z=\infty$ is specified, a third integral must assume a given value, otherwise this relation fixes v_0. Three more integrals must vanish to assure that $z(u)$ is single valued and that the profiles found in the z-plane are closed. *J. H. Giese* (Aberdeen, Md.).

Madhava Rao, B. S. Virial problems related to simple wing profiles. Proc. Indian Acad. Sci. Sect. A. **43** (1956), 53–66.

A well-known formula of Blasius for steady, incompressible, irrotational, plane flow gives the moment about the origin of coordinates as the real part of a contour integral. The imaginary part is called the virial about the origin. The Hamilton Centre of a profile is that point about which both the moment and virial vanish. The author believes that this point has some significance with regard to the stability of the profile "in the case of a real disturbance under flying conditions," and therefore works out a number of theorems regarding its geometry. These are applied to special families of profiles. *W. R. Sears.*

Carrier, G. F. On diffusive convection in tubes. Quart. Appl. Math. **14** (1956), 108–112.

In this note, the author investigates the problem of the time dependent solute content in a fluid flowing steadily at small Reynolds number through a long tube. The flow rate is assumed to be laminar and time independent. The boundary condition of the solute concentration s at the tube entrance is taken to be simple harmonic in time (i.e. equal to $\exp(i\Omega t)$). The solution of s sought by the author is of the expansion form which converges rapidly for small values of the parameter $\omega=\Omega r_0^2/\nu$ (say, $\omega<5$), where r_0 is the tube radius and ν the diffusivity. (This range of interest is encountered in experimentation concerning the salinity of ground water in permeable islands.) The author further compares his analysis with

that of G. I. Taylor [Proc. Roy. Soc. London. Ser. A. **219** (1953), 186–203] which was introduced in the treatment of a similar problem. *T. Y. Wu* (Pasadena, Calif.).

Kolosovskaya, A. K.; and Ickovič, I. A. The spatial problem of flow of an ideal fluid about porous obstructions. Kišinev. Gos. Univ. Uč. Zap. **11** (1954), 29–47. (Russian)

The authors consider steady irrational fluid flow about and through a porous membrane. The flow is irrotational outside and inside of a surface and the two flows are connected by the assumptions that the normal component of flow is the same on both sides of the surface and the change in pressure across the surface is a function of the flow across the surface and of a parameter λ whose physical meaning is not made clear. The problem is reduced to the solution of a pair of simultaneous nonlinear singular integral equations of the form

$$\left(\frac{\partial\varphi}{\partial q_1}\right)_S = -\frac{1}{4\pi}\iint_S \left(\frac{\partial G}{\partial q_1}\right)_S \cdot \Phi\left[\left(\frac{\partial\varphi}{\partial q_1'}\right)_S, \left(\frac{\partial\varphi}{\partial q_2'}\right)_S, \lambda\right] \partial S + \left(\frac{\partial\varphi^*}{\partial q_1}\right)_S$$

where the derivatives are evaluated on and the integral taken over the surface S; the functions G, Φ, φ^* are known; φ is the velocity potential, and q_1, q_1', q_2' are coordinates on S. If the flow is axially symmetric, then the pair can be replaced by a single equation of the same form. This problem can be solved by successive approximations for small values of λ. The proofs make use of the space of all Holder functions. *D. C. Kleinecke.*

Rheinboldt, Werner. Zur Berechnung stationärer Grenzschichten bei kontinuierlicher Absaugung mit unstetig veränderlicher Absaugegeschwindigkeit. J. Rational Mech. Anal. **5** (1956), 539–604.

The author investigates the incompressible laminar boundary layer with distributed surface suction in the case of arbitrary free-stream pressure gradient and arbitrary piecewise continuous suction velocity. The flow is assumed to satisfy the usual boundary-layer equations. Much of the paper is concerned with a rigorous mathematical analysis of the behaviour of the solution in the vicinity of a point of discontinuity of the suction velocity. In particular, after a suitable transformation of variables, it is shown that immediately downstream of such a point two series representations of the stream function can be found, one valid for small distances from the surface, the other for large distances.

The results of this lengthy analysis are summarized for purposes of practical applications, and tables of certain universal functions are presented. A relatively straightforward calculation method based on the results allows the solution to be continued a short distance beyond a discontinuity point of the suction velocity, if the initial velocity profile at this point is known. Farther downstream, where the variation of the solution is not as rapid, any of the standard numerical procedures can be used. The author suggests an improved version of Görtler's difference process for this purpose [Ing.-Arch. **16** (1948), 173–187; MR **10**, 336]. The solution can then be continued to the next discontinuity point, and the final velocity profile so obtained used as the initial velocity profile for a repetition of the whole process. Numerical examples are presented for the boundary layers on a flat plate and a circular cylinder in several cases involving suction. The results are given graphically, and include velocity distributions and curves showing the variation of displacement thickness and wall shear stress with distance along the wall. *D. W. Dunn* (Baltimore, Md.).

Davies, T. V. The forced flow due to heating of a rotating liquid. Philos. Trans. Roy. Soc. London. Ser. A. **249** (1956), 27–64.

This is a study of the stability of the forced flow of a fluid in a heated rotating cylindrical vessel. The basic flow, which is maintained by a temperature difference between the inner and outer boundaries of the vessel, is predominantly symmetric about the central axis and when instability arises a finite amplitude wave pattern appears, which progresses slowly relative to the rotating vessel. The method of small perturbations is employed to derive a stability equation and this is solved for two cases; barotropic flow, that is flow in which the isobaric and isothermal surfaces are coincident, and secondly a modified baroclinic flow. The results are compared with experimental data and the author concludes that the barotropic theory is inadequate to explain the stability of the system, and that a baroclinic theory is necessary. A relation between the mean vertical and horizontal temperature gradients in the fluid and the angular velocity of the system is shown to exist in the symmetric flow regime. *M. H. Rogers* (Urbana, Ill.).

Morawetz, Cathleen S. On the non-existence of continuous transonic flows past profiles. I. Comm. Pure Appl. Math. **9** (1956), 45–68.

C'est sans doute la première démonstration rigoureuse d'un résultat que l'on tenait pour vraisemblable depuis longtemps. On part d'un écoulement d'un fluide parfait autour d'un profil, subsonique à l'infini, mais supersonique dans une région contigue au profil. On sait que l'on peut effectivement construire de tels écoulements par la méthode de l'hodographe. Envisageons un profil voisin, la perturbation ayant le caractère général désirable. Existe-t-il un écoulement continu voisin. L'auteur montre que la réponse est négative. Le profil est supposé symétrique, l'écoulement sans circulation; on suppose que l'écoulement donné fait partie d'une famille d'écoulements continus, le potentiel et les composantes de la vitesse étant différentiables par rapport au paramètre définissant un profil de la famille; la famille dépend ellemême d'une fonction arbitraire. On montre que sauf pour un choix particulier de cette fonction, les vitesses ne peuvent rester continues. La méthode consiste à se ramener à un problème d'unicité pour un système linéaire d'équations aux dériviées partielles du type mixte, en linéarisant en quelque sorte le problème dans le plan de l'hodographe. Ce théorème d'unicité est démontré à l'aide de la méthode ABC de Friedrichs. La démonstration requiert des estimations difficiles et nombre de résultats partiels qui doivent être précisés, tels ceux relatifs au comportement de la solution à l'infini ou près d'un point d'arrêt. *P. Germain* (Paris).

Schäfer, Manfred. Über die stetige Rückkehr gestörter Überschallströmungen in den Unterschallbereich bei gemischten Strömungsfeldern. J. Rational Mech. Anal. **5** (1956), 217–250.

Cet article s'attaque à un problème voisin de celui étudié dans le travail analysé ci-dessus, mais apparemment l'auteur est conduit à des conclusions opposées. Utilisant la méthode de l'hodographe et la technique d'in-

tégration précédemment exposée [J. Rational Mech. Anal. 2 (1953) 383–412; MR 15, 176] l'auteur étudie l'existence d'écoulements continus voisins d'un écoulement transsonique donné. Après avoir développé les méthodes générales d'un tel calcul, l'écoulement circulaire autour d'un cercle (provenant d'un tourbillon unique placé au centre) est étudié en grand détail et avec beaucoup de soin; on met en particulier en évidence les „valeurs propres" pour lesquelles des difficultés apparaissent dans la construction de la solution perturbée. Un second exemple est traité, relatif à la perturbation d'un écoulement continu, d'ailleurs classique, dans lequel on observe une compression sans choc, c'est à dire un passage continu du supersonique au subsonique. Dans des conditions qui paraissent assez générales, on conclut encore à l'existence d'écoulements voisins continus. L'auteur fait d'ailleurs une critique des travaux de Guderley et de Busemann dont les conclusions diffèrent des siennes. Il s'agit plutôt de l'étude de quelques exemples que de l'énoncé d'un théorème précis, aussi est il difficile d'apprécier la portée du résultat qui nous est ici présenté. *P. Germain.*

Korobeĭnikov, V. P. On integrals of the equations of unsteady adiabatic motion of a gas. Dokl. Akad. Nauk SSSR (N.S.) 104 (1955), 509–512. (Russian)

The work of Lidov [same Dokl. (N.S.) 103 (1955), 35–36; MR 17, 207] and Sedov [Similarity and dimensional methods in mechanics, 3rd ed., Gostehizdat, Moscow, 1954; for a review of the ... ed. see MR 14, 809] on self similar motion of a gas with plane, cylindrical or spherical symmetry is extended. Use is made of dimensionless physical variables.

$$u=b^{1/m}t^{-n/m-1}V,$$
$$\varrho=ab^{-(k+3)/m}t^{n(k+3)/m-s}R,$$
$$p=ab^{-(k+1)/m}t^{n(k+1)/m-(s+2)}P,$$

where V, R, and P are functions of time t, a, b are parameters of mixed dimensions and the indices k, m, n, s are to be fixed. Series solutions

$$V=\sum V_i\tau^i,\ R=\sum R_i\tau^i,\ P=\sum P_i\tau^i$$

in powers of a modified time variable τ are sought by considering appropriate volume integrals of $\Phi_i(s)$ a general function of the entropy. In the case of perfect gas

$$\Phi_i(S)=\left(\frac{p}{\varrho^\gamma}\right)^m,$$

the coefficients in the series expansions for $i=0$ correspond to self similar motion, those for $i=1$ correspond to linear perturbation of self similar motion. Both cases were considered earlier by Lidov. Further terms evaluated here define higher order perturbations. *M. Holt.*

Keune, Friedrich; und Oswatitsch, Klaus. Äquivalenzsatz, Ähnlichkeitssätze für schallnahe Geschwindigkeiten und Widerstand nicht angestellter Körper kleiner Spannweite. Z. Angew. Math. Phys. 7 (1956), 40–63.

In transonic flow the velocity disturbance on low aspect ratio wings can be reduced to the velocity disturbance on bodies of revolution with the same distribution of cross-sectional area. From the similarity law for bodies of revolution corresponding similarity laws for low aspect ratio wings are deduced. The law of equivalence for the pressure drag is also given for bodies of revolution and pointed low aspect ratio wings with any cross-section at the trailing edge, provided the trailing edge is sufficiently far in the supersonic flow. This law is an extension to transonic flow of a formula of Ward [Quart. J. Mech. Appl. Math. 2 (1949), 75–97; MR 10, 644] originally valid for linearized supersonic theory only. (From author's summary.) *G. N. Lance* (Southampton).

★ **Diaz, J. B.; and Ludford, G. S. S. A transonic approximation.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 651–658. American Society of Mechanical Engineers, New York, 1955. $9.00.

En choisissant convenablement l'équation d'état d'un fluide fictif, liant la pression et la densité, l'étude des écoulements bidimensionnels d'un fluide compressible est ramené à l'intégration d'une équation aux dérivées partielles du type mixte analogue à celle de Tricomi, en ce sens qu'on peut écrire explicitement la solution générale du problème de Cauchy dont les données sont portées par la ligne sonique. Comme dans le cas de l'équation de Tricomi, les fonctions intervenant dans l'écriture de cette solution sont des fonctions de Bessel. Les auteurs discutent du meilleur choix possible des constantes numériques en vue d'approcher au mieux les propriétés du gaz réel; ils parviennent à établir ainsi une identité pratique entre leur fluide fictif et le fluide réel pour les régimes supersoniques; dans les régimes transsoniques l'approximation proposée est nettement supérieure à celle de Tricomi; pour les faibles vitesses, le résultat n'est pas entièrement satisfaisant. Aucun écoulement n'a été calculé par cette méthode, qui semble pourtant devoir être intéressante pour l'étude des écoulements transsoniques et supersoniques. *P. Germain* (Paris).

Meecham, William C. Fourier transform method for the treatment of the problem of the reflection of radiation from irregular surfaces. J. Acoust. Soc. Amer. 28 (1956), 370–377.

The title problem is formulated for the boundary condition of vanishing wave function by constructing an integral equation (Weber's solution to the wave equation) and expanding the kernel in even powers of the departure of the surface from a plane. A formal, iterative solution then is obtained by Fourier inversion, but (to the reviewer) only the first approximation — neglecting squares of (A) the surface slope (or incident wavelength/surface wavelength) and (B) surface amplitude/wavelength — is practically useful. The author correctly states that this first approximation is superior to those of "physical optics" with respect to (A) but fails to note that it may be inferior with respect to (B) and that it is equivalent with respect to both (A) and (B) to the "perturbation treatments" of Rice [Comm. Pure Appl. Math. 4 (1951), 351–378; MR 13, 605] and the reviewer [J. Acoust. Soc. Amer. 26 (1954), 191–199; MR 15, 840]. He also states that his formulation cannot be modified to meet the boundary condition of vanishing normal derivative of the wave function, but with this the reviewer (although failing to perceive the advantage of such a formulation, at least in first approximation) cannot agree. The author applies his first approximation to the problem of reflection of a plane wave from a sinusoidal surface and compares the result with his extension to oblique incidence of the result obtained by Rayleigh [Theory of sound, v. 2, 2nd ed., Macmillan, London, 1896, p. 89] for normal incidence and states that the latter neglects the first power of incident surface wavelength [the reviewer regards this comparison as

gratuitous, since Rayleigh showed (ibid, p. 93) that his result for normal incidence or "zero order" reflection at oblique incidence (ibid, p. 96) neglected only second powers of the wavelength ration, and no a priori, analytical superiority of the author's approximation is evident]; comparison also is made with experiment, with the author's first approximation proving superior to Rayleigh's first approximation [but, again, the author makes no reference to Rayleigh's improved approximation (ibid, p. 93), which would appear to be a priori superior to that of the author]. *J. W. Miles* (Los Angeles, Calif.).

Wu, T. Yao-tsu. Small perturbations in the unsteady flow of a compressible, viscous and heat-conducting fluid. J. Math. Phys. **35** (1956), 13–27.

L'auteur étudie les mouvements d'un fluide compressible, visqueux et conducteur thermique; le fluide est soumis à des forces extérieures et à des actions thermiques. On se place dans le cas où les équations du mouvement peuvent être linéarisées. Le principe de la superposition des solutions permet de décomposer celles-ci en ondes transversales pour lesquelles la divergence de la vitesse est nulle et en ondes longitudinales pour lesquelles le rotationnel de la vitesse est nul. Ces dernières sont étudiées dans le cas des mouvements rectilignes au moyen de la transformation de Laplace; des formules asymptotiques sont données pour les petites et les grandes valeurs du temps. *H. Cabannes* (Québéc, P.Q.).

★ **Oguchi, Hakuro. On the reflected wave in Mach reflection.** Proceedings of the First Japan National Congress for Applied Mechanics, 1951, pp. 353–358. Science Council of Japan, Tokyo, 1952.

The diffraction of a plane shock by a corner of small angle δ in a straight wall is considered. The treatment continues Lighthill's first order theory [Proc. Roy. Soc. London. Ser. A. **198** (1949), 454–470; MR **11**, 478] so that terms $O(\delta)$ can be included. It is established that the reflected wave is a shock of strength $O(\delta^2)$ which degenerates into a sound wave at the triple point. In both subsonic and supersonic cases the reflected shock has maximum strength near its point of contact with the tangent from the corner. *M. Holt* (Providence, R.I.).

Sambasiva Rao, P. Supersonic bangs. I. Aero. Quart. **7** (1956), 21–44.

L'auteur étudie les détonations provoquées par un obstacle animé d'un mouvement de translation non uniforme de vitesse supersonique. La théorie linéaire est abandonnée; elle ne peut donner une description satisfaisante des phenomènes de choc dont les propriétés sont essentiellement non linéaires. Le comportement du choc à l'infini est étudié à l'aide d'une extension de la théorie de Witham [Comm. Pure Appl. Math. **5** (1952), 301–348; MR **14**, 330]; d'autre part, dans le cas d'un mouvement de révolution, la courbure du choc sur l'axe est calculée, pour un obstacle dont le demi-angle au sommet est petit; la courbure du choc est plus influencée par l'accélération de l'obstacle que par la courbure de l'obstacle. *H. Cabannes* (Marseilles).

Sambasiva Rao, P. Supersonic bangs. II. Aero. Quart. **7** (1956), 135–155.

Dans cette seconde partie l'auteur reprend le problème précedent dans le cas d'un obstacle dont le trajectoire est curviligne. Le paramètre fondamental qui apparaît est la composante de l'accélération suivant les directions qui font avec la vitesse l'angle arccos M^{-1}, où M désigne le nombre de Mach. En dehors de la modification de cette composante, qui résulte de la courbure de la trajectoire, les principaux résultats obtenus dans la première partie se retrouvent sous une forme analogue. *H. Cabannes.*

★ **Oguchi, Hakuro. The flow behind an attached curved shock.** Proceedings of the Third Japan National Congress for Applied Mechanics, 1953, pp. 243–246. Science Council of Japan, Tokyo, 1954.

The flow near the nose of a curved profile differing slightly from a straight wedge in uniform supersonic flow is considered. The technique used is developed from Lighthill's treatment of anisentropic linearised flow [Phil. Mag. (7) **40** (1949), 214–220; MR **10**, 641] and is applied to cases of both supersonic and subsonic flow behind the shock. The reflection factor at the shock is calculated and its variation along the shock agrees with earlier estimates. *M. Holt* (Providence, R.I.).

Germain, Paul; et Gundersen, Roy. Sur les écoulements unidimensionnels d'un fluide parfait à entropie faiblement variable. C. R. Acad. Sci. Paris **241** (1955), 925–927.

Linear perturbations of one dimensional isentropic unsteady flow are considered. The equations governing the perturbed flow are linear but non-homogeneous, due to the presence of first order entropy terms. Explicit solutions are given when the basic flow is uniform or a simple wave. *M. Holt* (Providence, R.I.).

Botta, A. Une méthode simple de calcul de l'absorption du son par viscosité seule, dans des fluides homogènes limités. Bull. Tech. Suisse Romande **82** (1956), 170–173.

Išlins'kiĭ, O. Yu. On electro-simulation of river flows. Dopovidi Akad. Nauk Ukrain. RSR. **1956**, 124–126. (Ukrainian. Russian summary)

Vasil'ev, V. A. On the form of a mound of ground water between two drains in a waterproof support in the presence of infiltration. Prikl. Mat. Meh. **19** (1955), 106–108. (Russian)

The author obtains the hodograph velocity distribution of the present problem in terms of elliptic functions. *K. Bhagwandin* (Oslo).

Belyakova, V. K. Unsteady flow of ground water to a horizontal drain. Prikl. Mat. Meh. **19** (1955), 234–239. (Russian)

The author studies the problem of the unsteady motion of ground-water in the presence of one or more drains in homogeneous or inhomogeneous grounds. The analysis proceeds along well-known lines. She reduces the problem to a study of the second-order Volterra integral equation. Graphs are plotted for the free-surface function for different values of the parameters involved. *K. Bhagwandin.*

Yalin, Selim. Über die Sickerströmung in einem stetig heterogenen Raum. Bull. Tech. Univ. Istanbul **7** (1954), 59–78. (Turkish summary)

The author's study (Darcy-type) of infiltration in heterogeneous media [cf. P. Ya. Polubarinova-Kočina, Theory of motion of ground water, Gostehizdat, Moscow, 1952; MR **15**, 71; Litwiniszyn, Ann. Soc. Polon. Math. **22** (1949), 185–194; MR **11**, 699] is based on the study of

the equation $\partial^2H/\partial x^2+\partial^2H/\partial y^2+y^{-1}\partial H/\partial y=0$ (H denotes the 'hydraulic load-function'). This equation is found on the hypothesis that the filtration-coefficient $k(x, y, z)$ is expressible as $k_x(x)\cdot k_y(y)\cdot k_z(z)$, (the filtration falls off with depth). As an example, the author obtains a solution to infiltration beneath a finite dam. The expression for the velocity components, the flux, the pressure exerted against the lower part of the dam, the modulus of the curl are given in terms of Bessel and exponential functions.

The reading of the author's paper is made difficult on account of the numerous printing errors, and badly reproduced figures and tables. *K. Bhagwandin.*

See also: Fujikawa, p. 1089; Kawaguti, p. 1091; Moreau, p. 1166; Pham Mau Quan, p. 1143; Teodorescu, p. 1161; Tomotika and Yosinobu, p. 1091.

Elasticity, Plasticity

Rozovskiĭ, M. I. On nonlinear equations of creep and relaxation of materials for a composite stressed state. Ž. Tehn. Fiz. **25** (1955), 2339–2355. (Russian)

The author has formulated general non-linear visco-elastic stress-strain equations. The non-linearity is due to (i) non-linear instantaneous elasticity, and (ii) relaxation time dependent on the dilatation. In this paper he shows that the equations reduce in special cases to those for a Mises solid in plasticity, to those for ageing, to those for creep etc. Finally he considers the semi-infinite plane acted on by a line force and the sphere under internal pressure for different forms of the stress-strain equations.

D. R. Bland (London).

★**Ržanicyn, A. R. Ustoĭčivost' ravnovesiya uprugih sistem. [Stability of equilibrium of elastic systems.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 475 pp. 15.30 rubles.

This volume contains a systematic presentation of the basic problems in the field of stability of simple engineering structures. It begins with an analysis of stability of elastic structures with a finite number of degrees of freedom and concludes with a brief chapter on stability of elastic plates.

The book should prove of interest to structural designers in the West because it includes a connected account of the contribution to the problem of stability of rods made by the Soviet investigators. Graphical methods of determining stability are given considerable prominence.

The contents of the book and its scope are indicated by the following list of chapter headings. 1. Stability of systems with one degree of freedom (pp. 13–24). 2. Systems with several degrees of freedom (pp. 25–46). 3. Non-homogeneous problems of systems with one degree of freedom (pp. 47–76). 4. Non-homogeneous problems of stability of systems with several degrees of freedom (pp. 77–87). 5. Examples of stability of systems with finite number of degrees of freedom (pp. 88–107). 6. Stability of compressed prismatic elastic rod (pp. 108–121). 7. More refined solutions of the problem of stability of an elastic prismatic rod (pp. 125–141). 8. The fundamental differential equation of the problem of stability of a compressed elastic rod (pp. 142–168). 9. Further solution of the problem of stability of a straight elastic rod (pp. 169–178). 10. The method of initial parameters in the calculation of stability (pp. 179–209). 11. Compressed laterally loaded elastic rods (pp. 210–233). 12. Graphical solutions of the problem of a compressed laterally loaded rod (pp. 234–261). 13. Stability of rods with varying cross-sections (pp. 262–306). 14. Stability of rods beyond proportional limit (pp. 307–325). 15. Stability of compressed laterally loaded rods beyond proportional limit (pp. 326–344). 16. Stability of rods on an elastic foundation (pp. 345–365). 17. Stability of composite rods (pp. 366–388). 18. Compressed laterally loaded composite rods (pp. 389–409). 19. Stability of composite elastically bound rods and other types of composite rods (pp. 410–420). 20. Stability of curved rods (pp. 421–441). 21. Stability of I-beams (pp. 442–457. 22. Stability of plates (pp. 458–468).

The demands on the mathematical equipment of the reader are not great and the reader familiar with Timoshenko's "Theory of elastic stability" [McGraw-Hill, New York, 1936] should be able to read this book with profit.

I. S. Sokolnikoff (Los Angeles, Calif.).

★**Gol'denblat, I. I. Nekotorye voprosy mehaniki deformiruemyh sred. [Some questions of the mechanics of deformable media.]** Gosudarstv. Izdat. Tehn.-Teor. Lit., Moscow, 1955. 271 pp. 9.55 rubles.

The first two chapters (70 pp.) conform with a practice which almost characterizes authors in finite elasticity — to make their treatment self-contained by including an abridgment of the excellent exposition of the basic notions of finite deformation left by the Cosserat brothers [Ann. Fac. Sci. Toulouse **10** (1896), I.1–I.116]. It is a formidable task to improve the work of masters.

The non-resumé portion of the book begins most interestingly in chapter III, The Joint Invariants of Deformation and Stress Fields. This material first appeared in the author's earlier book [Some questions of the theory of elastic and plastic deformations, Gostroiizdat, Moscow, 1950]. In this connection, the author cites the book of G. B. Gurevich [Foundations of the theory of algebraic invariants, OGIZ, Moscow-Leningrad, 1948; MR **11**, 413]. As further evidence that the formulation of stress-deformation relations via a scrupulous observance of classical algebraic invariant theory seems to be an idea whose time has come, we remark that R. S. Rivlin and J. L. Ericksen, while working together at NRL (1952–54), were investigating this same subject [J. Rational Mech. Anal. **4** (1955), 323–425, 681–702; MR **16**, 881; **17**, 210]. In formulating stress-deformation relations for small displacements of a non-linear elastic material, both isotropic and anisotropic, the author presents a single technique which branches to yield a) the classical result for a linear elastic material, b) the result for a plastic material while loading, c) the result for a strain-hardening material while loading.

The last seven chapters (110 pp.) are concerned with the thermodynamics of deformation, a subject in which the author is particularly well versed. Amidst an abundance of thought-provoking material are to be found invariantive, non-linear, first order partial differential equations for the various thermodynamic potentials and differentials of these same potentials which might seemingly be applicable to current problems in plasma physics. *T. C. Doyle* (Los Alamos, N.M.).

Varvak, P. M. On the strain function. Akad. Nauk Ukrain. RSR. Prikl. Mat. **1** (1955), 479–480. (Ukrainian. Russian summary)

Sandro Dei Poli in "Costruzioni metalliche" (1953)

introduced for plane elasticity problems a strain function which satisfies the equilibrium and compatibility conditions. The author of this paper shows that the conventional Airy stress function differs from the Sandro Dei Poli strain function by a constant factor. He constructs another strain function different from the one of Sandro Dei Poli which also equals the Airy stress function multiplied by a constant factor. The author implies without proving it rigorously that any strain function which satisfies the equilibrium and compatibility conditions must differ from the stress function by a constant factor.

T. Leser (Aberdeen, Md.).

Trifan, D. Stress theory of plastic flow. J. Math. Phys. 35 (1956), 44–52.

The typical stress-strain law for a work-hardening plastic material gives the strain increment as a function of the instantaneous stress and the increment of stress. In contrast to this "strain theory" of plastic flow, the author considers a "stress theory" that lets the stress increment depend on the instantaneous strain and the strain increment. Typical stress-strain laws of this type are discussed, and two absolute minimum principles associated with this theory of plasticity are established.

W. Prager (Providence, R.I.).

Thomas, T. Y. Characteristic surfaces in the Prandtl-Reuss plasticity theory. J. Rational Mech. Anal. 5 (1956), 251–262.

The author considers the Prandtl-Reuss equations of plasticity, modified so as to be dynamically correct. He shows that the yield condition is satisfied throughout a flow if it is satisfied on a surface intersected by the trajectory of each particle. The characteristic equation is derived and given in invariant form. It is shown that the same equation holds on certain surfaces of discontinuity, it being assumed in both cases that the surface is not a material surface. He indicates that it is, in general, impossible for surfaces of discontinuity of the order considered to be material surfaces. *J. L. Ericksen.*

Truesdell, C. Hypo-elastic shear. J. Appl. Phys. 27 (1956), 441–447.

The theory of hypo-elasticity proposed by Truesdell [J. Rational Mech. Anal. 4 (1955), 83, 983] is a smooth theory of elastic response based on time rates which, for small strains, agrees with the classical linear theory of elasticity. For large strains the motion actually experienced by the specimen under test must be found since the basic relations connect only rates. In any given problem there will be a relation between stress and strain but this is not known in advance and will depend on the manner in which the specimen is loaded. In part I of the paper simple shear is discussed for a hypo-elastic body of grade one. Such bodies may soften or harden in shear, depending on the value of dimensionless constant which has no effect when the strain is small. For bodies which soften, a theoretical prediction of "hypo-elastic yield" is obtained.

Part II concerns a type of hypo-elastic body for which yield of the von Mises type appears to follow if the stress intensity is sufficiently great. The equations are solved for simple shear. It is shown that if von Mises yield occurs, hypo-elastic yield must occur at a smaller stress. For large values of a certain parameter, von Mises yield is imaginary and only hypo-elastic yield occurs. For moderate values of the parameter, hypo-elastic yield appears as primary yield, with von Mises yield at infinite strain. For small values of the parameter, hypo-elastic yield and von Mises yield are indistinguishable, and the stress-strain curve is similar to the idealized forms assumed in the conventional Prandtl-Reuss theory of plasticity. *A. E. Green* (Newcastle-upon-Tyne).

Gran Olsson, R. Über zwei klassische Probleme aus der Theorie der ebenen Elastizität. Norske Vid. Selsk. Forh., Trondheim 28 (1955), 161–165 (1956).

Derivation and discussion of the state of displacement associated with an Airy function F which depends on the radial coordinate r, only. The known solution $F=c\phi$ is also discussed. *E. Reissner* (Cambridge, Mass.).

Bowie, O. L. Analysis of an infinite plate containing radial cracks originating at the boundary of an internal circular hole. J. Math. Phys. 35 (1956), 60–71.

Following Muschelisvili's approach, the region indicated by the title is mapped on the exterior of the unit circle and the mapping function replaced by a polynomial approximation with cusps corresponding to the crack roots. The complex stress functions are found for the cases of no forces on the internal boundary and either hydrostatic tension or tension in one direction only at infinity. The change in strain energy due to the crack and the critical load according to the Griffith theory for failure are then calculated. Numerical results are tabulated for 2 radial cracks, with hydrostatic or simple tension at infinity. *R. C. T. Smith* (Cambridge, Mass.).

★**Parkus, Henry. Stress in a centrally heated disk.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 307–311. American Society of Mechanical Engineers, New York, 1955. $9.00.

Discussion is given of a thermal problem for an infinite, thin slab of homogeneous, isotropic, elastic material. Initially, the slab is at the uniform temperature of the surrounding medium, say air. Suddenly, heat is supplied to the centre of the slab at a constant rate maintained over some finite period of time. The conduction of heat outwards into the slab and also from the slab into the air causes thermal stresses and displacements to be set up. Here, this problem is treated as a two-dimensional one, and any temperature variation of thermal and mechanical properties is entirely neglected. The determination of the temperature field is straightforward, and this is accomplished through the use of Laplace transform techniques. The complete determination of the stress and displacement field is complicated by inertial and plastic effects. Apart from some general remarks on the 'cooling period', attention is mainly confined to the 'heating period'. A solution in which plastic effects are neglected is first found, use again being made of Laplace transform techniques. Analysis is then given for the case when plastic effects are included and inertial effects are neglected. The Prandtl-Reuss equations are adopted, and the solution of the problem is of course now complicated by the presence of an outwards-moving elastic-plastic boundary. The reviewer has two criticisms of this latter part of the author's analysis. First, it may well not be realistic to neglect the inertial effects in the plastic region. Second, although the state of stress in the plastic region is 'statically-determinate' [see the general solution

given by R. Hill, The mathematical theory of plasticity Oxford, 1950, p. 308; MR 12, 303], this state is not necessarily the very simple one assumed here in which the radial and circumferential stresses are numerically equal to the yield stress in tension. Further studies of the present problem would be valuable, and some simplifications would probably accrue from the adoption of the Tresca yield condition and associated flow rule.

H. G. Hopkins (Sevenoaks).

Tolokonnikov, L. A. Finite deformations for pure bending of a rectangular strip. Inžen. Sb. 22 (1955), 188–192. (Russian)

The author analyzes the finite deformations of a rectangular strip generated by bending moments. Starting from some simplifying hypotheses he derives directly the equations of compatibility for deformations and under the assumption of a general plane stress state he determines the equation of equilibrium of the element considered. From this he deduces a differential equation which makes it possible to avoid the calculations of finite deformations through displacements. An example, completely developed, is given. *T. P. Anđelić* (Belgrade).

Lehnickiĭ, S. G. Some cases of elastic equilibrium of an anisotropic plate with a noncircular opening (plane case). Inžen. Sb. 22 (1955), 160–187. (Russian)

The author has in a previous paper [Inžen. Sb. 17 (1953), 3–28; MR 16, 540] developed a method for an approximative solution of plane elastic case relative to an anisotropic plate, weakened through an opening slightly different from the circular form. He has analyzed the distribution of stresses in the neighbourhood of an opening with four axes of symmetry in the cases of the extension and of bending generated through moments.

In the present paper he applies this method to solution of the plane case for an anisotropic plate with an opening slightly different from the elliptic form. He gives the approximative solutions for the case of an opening in an orthotropic plate which is near to an equilateral triangle with rounded off corners, and to an oval opening (more exactly, to an opening of the rectangular form with curved shorter sides). The general case of distribution of external forces is considered as well as two particular cases: of extension and of bending through moments.

T. P. Anđelić (Belgrade).

Adkins, J. E. Associated problems in two-dimensional elasticity. J. Mech. Phys. Solids 4 (1956), 199–205.

In the standard complex variable treatment of generalized plane stress and plane strain the problem is reduced to finding $\Omega(z, \alpha)$, $\omega(z, \alpha)$ given the values of

$$\Phi_\alpha = \alpha\Omega(z) + z\overline{\Omega}'(\bar{z}) + \overline{\omega}'(\bar{z})$$

on the boundary ($\alpha=1$ or $\alpha=-\kappa$). Φ_1 gives the stress resultant over a curve from some fixed point to z and $\Phi_{-\kappa}$ the displacement at z. For a single boundary the two problems (boundary-stress given, boundary-displacement given) can be solved simultaneously by regarding α as a variable parameter. The solution is expressed neatly in terms of the two parameter function

$$\Phi_{\beta,\alpha} = \beta\Omega(z, \alpha) + z\overline{\Omega}'(\bar{z}, \alpha) + \overline{\omega}'(\bar{z}, \alpha).$$

The generalization to several boundaries is given and several applications discussed. *R. C. T. Smith.*

★ **Horvay, G. Thermal stresses in rectangular strips.** Proceedings of the Second U. S. National Congress of Applied Mechanics, Ann Arbor, 1954, pp. 313–322. American Society of Mechanical Engineers, New York, 1955. $9.00.

This paper is a study of a thermal problem of plane stress for an infinite rectangular strip of homogeneous, isotropic, elastic material. The left- and right-hand halves of the strip are at different temperatures but otherwise the temperature field is uniform and it is also steady. The formal solution of the problem is straightforward. However, the consequent calculation of numerical results is not so straightforward, and the author is primarily concerned with this aspect of the problem, three lines of approach being considered. The exact solution to the problem is first found in terms of Fourier integrals. In this, and cognate problems, the analysis should be developed in such a way that divergent integrals are altogether avoided [see, Hopkins, Proc. Cambridge Philos. Soc. 46 (1950), 164–181; MR 11, 285]. In this respect, the present analysis is somewhat lacking in mathematical precision. Except in the immediate neighbourhood of the temperature discontinuity, numerical values of the stresses and displacements, given directly in terms of Fourier integrals, are not obtained without excessive computation. An alternative procedure is to base the numerical work on series expansions for the Fourier integrals, these expansions being obtained through the use of the calculus of residues. This corresponds to the representation of the stress function in terms of Fadle's biharmonic functions. In this case, the computation is now most severe in the immediate neighbourhood of the temperature discontinuity, and this situation is due to Gibbs' phenomenon. As a complete alternative to the derivation of numerical results from the exact solution, an approximate solution is next derived. This is based on the 'method of selfequilibrating functions' previously developed by the author [see G. Horvay, J. Appl. Mech. 20 (1953), 87–94, discussion 576–582; MR 14, 819]. Essentially, in the practical application of this method, boundary conditions are satisfied exactly and strain compatibility conditions are only approximately satisfied. Here, the method is applied to obtain an approximate solution valid in only one half of the strip, the requisite boundary conditions at the temperature discontinuity being determined in an approximate manner from the Fourier integral solution. This procedure leads to formulae from which approximate numerical values of the stresses and displacements may be calculated with relative ease. Extensive numerical data are given in both tabular and graphical form. As expected, the most critical stress conditions occur where the edges of the strip cross the temperature discontinuity. *H. G. Hopkins* (Sevenoaks).

Trostel, R. Instationäre Wärmespannungen in Hohlzylindern mit Kreisringquerschnitt. Ing.-Arch. 24 (1956), 1–26.

This paper [a shortened version of a Dissertation, Tech. Univ. Berlin-Charlottenburg] is a rather general study of unsteady associated temperature and stress fields in hollow, circular, elastic cylinders under conditions of axial symmetry. The usual simplifying assumptions, homogeneity and isotropy of material and thermal independence of relevant physical constants, are adopted. In addition, the stress state is taken to be one of equilibrium, inertia terms in the stress equations then being neglected. Attention is given mainly to various classes of

problems, their formal solutions being derived in the usual way via standard analysis. One special problem is treated in some detail, and numerical results are obtained. This problem, one of plane strain, concerns a thick circular tube that is initially at a uniform temperature and is suddenly subjected to a particular type of temperature distribution (exponentially decaying in time to a steady value) over its inner surface. The boundary conditions at the outer surface of the tube correspond to either thermal insulation from or radiation into the surrounding medium.

H. G. Hopkins (Sevenoaks).

Arutyunyan, N. H.; and Čobanyan, K. S. On torsion of prismatic rods composed of different materials with account taken of creep. Akad. Nauk Armyan. SSR. Dokl. 21 (1955), 3–9. (Russian. Armenian summary)

The authors consider the problem of torsion of a prismatic rod with arbitrary cross-section D consisting of two regions D_1 and D_2. It is assumed that in region D_1 the material exhibits creep and in region D_2 the ,material is elastic. The problem is formulated in terms of two time-dependent stress functions F_1 and F_2, defined in regions D_1 and D_2 respectively, in an analogous manner to the formulation of the classical elastic torsion problem. The equations governing F_1, F_2 and the conditions on the functions on the lateral surface of the rod and the interface of the two regions are derived. The authors indicate that solutions to particular problems will appear in a later work.

R. T. Shield (Providence, R.I.).

Barta, J. Inequality relation between torsional and flexural rigidity. Acta Tech. Acad. Sci. Hungar. 14 (1956), 477–479. (Russian, French and German summaries)

Let $R_{f\max}$ and $R_{f\min}$ be the maximum and minimum values of the flexural rigidity of a beam of given cross section. It is known that these are related to the maximum and minimum moments of inertia of the cross section by $R_{f\max}=EI_{\max}$, $R_{f\min}=EI_{\min}$, where E is Young's modulus. Consequently, $R_{f\max}+R_{f\min}=EI_0$, where I_0 is the polar moment of inertia. The author observes that this fact, together with the inequality of Diaz and Weinstein [Amer. J. Math. 70 (1948), 107–116; MR 9, 480] for the torsional rigidity $R_t\leq EI_0/2(1+\nu)$, yields

$$R_t\leq(R_{f\max}+R_{f\min})/2(1+\nu)<R_{f\max}$$

when Poisson's ratio ν is positive. *H. F. Weinberger.*

De Schwarz, Maria Josepha. Effetti flessionali di carichi sui bordi trasversali di volte cilindriche circolari. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 20 (1956), 49–55.

Vernotte, Pierre. Le problème de la barre en régime périodique lorsque les propriétés physiques dépendent de la température. C. R. Acad. Sci. Paris 242 (1956), 2808–2810.

Hatiašvili, G. M. Elastic equilibrium of a composite cylindrical beam with a loaded lateral surface in the case of different Poisson coefficients. Soobšč. Akad. Nauk Gruzin. SSR 16 (1955), 19–25. (Russian)

An outline of a solution of the problem stated in the title, which depends on a solution of an auxiliary plane problem of elasticity is presented in this note. The author proposes to publish a detailed version later.

I. S. Sokolnikoff (Los Angeles, Calif.).

Schnell, W. Zur Berechnung der Beulwerte von längs- oder querversteiften rechteckigen Platten unter Drucklast. Z. Angew. Math. Mech. 36 (1956), 36–51. (English, French and Russian summaries)

Für eine isotrope Rechteckplatte der Dicke t mit den Abmessungen: a gleich Länge in der x-Richtung, b gleich Länge in der y-Richtung erhält man aus den Gleichgewichtsbedingungen an der verformten Mittelfläche und dem Elastizitätsgesetz die Differentialgleichung für die Durchbiegung:

$$N\Delta\Delta w+\sigma_1 t\frac{\partial^2 w}{\partial x^2}+\sigma_2 t\frac{\partial^2 w}{\partial y^2}=0,$$

wobei N die Plattensteifigkeit, $\sigma_1 t$ und $\sigma_2 t$ die Randlasten in x- und y-Richtung sind und zwar als Druck positiv gerechnet. Alle Abmessungen werden auf die Breite b bezogen, also $\xi=x/b$, $\eta=y/b$, $\alpha=a/b$ womit als Grundgleichung für die Ausbiegung erhalten wird: $\Delta\Delta w+\pi^2k(w''+p\ddot{w})=0$, wo Striche und Punkte Ableitungen nach ζ bzw. η bedeuten. Ferner bedeutet k die auf die Eulerspannung eines Druckstabes der Länge b bezogene kritische Spannung σ_E, deren Beulwert $k=\sigma_1/\sigma_E$ mit $\sigma_E=\pi^2N/b^2t$. Die kritische Spannung σ_2 wird auf σ_1 bezogen: $\sigma_2=p\sigma_1$, wo p eine Zahl bedeutet. Da die Matrizenmethode von einer gewöhnlichen Differentialgleichung ausgeht, können nur solche Randbedingungen zugelassen werden, die eine Überführung der partiellen Differentialgleichung in eine gewöhnliche gestatten. Es wurden unter anderen folgende Fälle untersucht: 1) Die einseitig gedrückte Platte mit Längssteifen. 2) Die einseitig gedrückte Platte mit Quersteifen. 3) Die beiderseitig gedrückte Platte mit Längssteifen, wobei sowohl zwei wie drei Steifen in Betracht gezogen werden. 4) Die unversteifte Platte mit einer längs des Randes veränderlichen Last, wobei die Fälle einer stückweise konstanten sowie einer linear veränderlichen Belastung besonders untersucht werden. [Auf diese Weise können die Beulwerte der reinen Biegung ermittelt werden.] 5) Endlich wird die Differentialgleichung sowie die Beulbedingung für dei orthogonale, anistrope (orthotrope) Platte unter beiderseitigen Druck aufgestellt und der zugehörige Beulwert ermittelt. Die Arbeit ist trotz ihrer relativen Kürze sehr inhaltsreich und liefert viele neue Gesichtspunkte zur Theorie und Anwendung der Beulung von Platten. [Druckfehler: Die vorletzte Gleichung auf S. 36 muss heissen: $\sum_i \lambda_i^2 C_i=v_0$". Auf S.37, Zeile 9 von unten muss es heissen: „isotrope Rechteckplatte"].

R. Gran Olsson (Trondheim).

Hu, Hai-chang. On the matrix theory of continuous beams on elastic foundation. Acta Sci. Sinica 2 (1953), 171–178.

Let A and B denote the ends of one span of a continuous beam resting in equilibrium on an elastic foundation. The ends A and B are hinged to rigid supports. When there is no load on the span the relations between the bending moments M and the rotations θ at A and B can be written in the matrix form

$$\begin{pmatrix}\theta_B\\ M_B\end{pmatrix}=\begin{pmatrix}-E & F\\ G & -E\end{pmatrix}\begin{pmatrix}\theta_A\\ M_B\end{pmatrix},$$

where the prescribed constants E, F and G are determined by the physical characteristics of the span and its foundation. An iterated application of that matrix formula gives a relation between the moments and rotations at the ends of a beam of n spans. When a unit couple is applied to one end which is hinged and when the other end of the beam

is elastically hinged ($M_n=\mu_n\theta_n$, $y_n=0$) the angles and moments at all supports can be found by elementary calculations. It is then shown how these results can be used to find the rotations and moments at the ends of the spans of loaded continuous beams of n spans resting on an elastic foundation when the ends of the spans are hinged to rigid supports except for one end of the beam which is elastically hinged. *R. V. Churchill* (Ann Arbor, Mich.).

Isida, Makoto. On the conditions of a free elliptic boundary in the two-dimensional problems of elasticity. Sci. Papers Fac. Engrg. Tokushima Univ. **1953**, no. 5, 35–42.

This paper gives the results for a thin infinite isotropic plate with an elliptic hole, under no external stress around the hole, in cartesian coordinates instead of elliptic coordinates. It is hoped to use these results in a later paper for the multiply-connected domains such as a semi-infinite plate or a strip with an elliptic hole.
R. M. Morris (Cardiff).

Marguerre, K. Vibration and stability problems of beams treated by matrices. J. Math. Phys. **35** (1956), 28–43.

In vibration problems, matrix calculus is known in connection with the method of influence coefficients. The method of Myklestad [J. Aero. Sci. **11** (1944), 153–162; MR **5**, 251] is in the author's opinion more effective than the influence coefficient method for determining the higher modes of vibration of a shaft of variable cross-section with intermediate supports (critical speed of machinery). The method of Myklestad, however, can also be interpreted as a matrix method [W. T. Thomson, J. Appl. Mech. **17** (1950), 337–339; MR **12**, 221], a fact which greatly facilitates understanding and handling of the procedure. The author therefore gives a short survey of the matrix method for the differential equation of a vibrating beam, and for related problems of a column. It is pointed out that the equation for the "eigenvalue" appears always as the condition that a second order subdeterminant of the governing matrix must vanish; in the influence coefficient method, however, the determinant is of the order of degrees of freedom. Systematic difficulties arise in the especially interesting cases of intermediate rigid supports or ideal pin connections. Even if the supports or the hinges are elastic and therefore can be taken into account by a transition matrix, another very inconvenient thing can occur: the coefficients of the equation for the "eigenvalues" appear as differences of very large numbers. Both difficulties can be overcome by introducing the notion of a kind of associated matrix, by the author called Δ-matrix, for which very simple rules of calculation can be given. Further details may be found in papers of Fuhrke [Ing. Arch. **23** (1955), 329–348] and Schnell, Z. Angew. Math. Mech. **35** (1955), 269–284; MR **17**, 319]. There are some misprints in the paper.
R. Gran Olsson (Trondheim).

Hayes, W. D.; and Miles, J. W. The free oscillations of a buckled panel. Quart. Appl. Math. **14** (1956), 19–26.

The subject of this paper is the oscillations of a two-dimensional buckled plate. The problem is of considerable practical interest and serves as a simple model of the phenomenon often described as "oil canning". Furthermore, it is of considerable theoretical interest that the original partial differential equation of motion is linear and that the deflection is required to be small, but that the problem is non-linear in consequence of the buckling constraint. The rather more difficult problem of the aerodynamic excitation of a buckled plate has been studied previously by the authors [W. D. Hayes, No. Amer. Avia. Rep. AL-1029, Downey, Calif. 1950; J. W. Miles, ibid. AL-1140], but the simplicity of the present problem allows a more complete solution. In particular, only free oscillations are treated, whereas self excited oscillations are possible under aerodynamic excitation. — The non-linear equations of motion of the two-dimensional buckled plate are established in dimensionless form. A trigonometric expansion in the space variable is introduced and an approximate solution is obtained for the period of free oscillation on the assumption of only two degrees of freedom, one of which is eliminated by the buckling constraint. The periods of the simplest asymmetric and symmetric modes are plotted as a function of the energy level. It is found that for very small energy levels the period of the buckled panel lies between the periods of the two unbuckled degrees of freedom. As the energy is increased the period approaches infinity at some critical energy level and thereafter decreases monotonically. *R. Gran Olsson* (Trondheim).

Gran Olsson, R. On the integration of the differential equation of thin elastic plates of variable thickness. Norske Vid. Selsk. Forh., Trondheim **28** (1955), 176–181 (1956).

Aggarwala, B. D. Singularly loaded rectilinear plates—bending by isolated couples. Bull. Calcutta Math. Soc. **47** (1955), 87–114.

This paper is a continuation of previous work [Z. Angew. Math. Mech. **34** (1954), 226–237; MR **15**, 1003] on problems of the small transverse displacements of thin elastic plates subjected to singular types of applied force. Attention is confined to simply-supported polygonal plates, and as before the analysis is based upon the theory of functions of a complex variable. The problem of an isolated couple is solved for some cases of square and triangular plates, the couple acting at a point on an axis of symmetry of the plate. The (closed) solutions are expressed in terms of elliptic functions. Some numerical results are given. *H. G. Hopkins* (Sevenoaks).

Levi, Beppo. Intorno al calcolo della inflessione delle piastre sottili. Univ. e Politec. Torino. Rend. Sem. Mat. **14** (1954–55), 67–74.

Informal discussion of the author's article with the same title. Math. Notae **12–13**, 79–193 (1954). (Spanish). MR **16**, 646.

Burmistrov, E. F. Computation of sloping orthotropic shells taking account of finite deformations. Inžen. Sb. **22** (1955), 83–97. (Russian)

The author considers an open orthotropic shell of small curvatures with a rectangular plane projection. The principal elastic directions are parallel to the boundaries. The distributed load is normal to the outside surface, and in the final stage assumed to be constant. The author considers two cases: a) when the supports prevent displacements at the boundaries, and b) when the supports allow displacements at the boundaries. Solving the first case the author represents both displacements and curvatures by Fourier series. The general solution giving the stress function contains terms some of which are single and some double series. The coefficients of these series are determined by a complicated system of equations also

containing terms with double series. To obtain comparatively simple results which can be used for numerical computations the author makes an approximation keeping only the first term in each series. When the curvatures are set equal zero this approximate solution coincides curiously enough with the solutions for a plate obtained by other authors [J. Prescott, Applied elasticity, Longmans-Green, London, 1924; V. M. Darevskii, Trans. Central Aero-Hydrodynam. Inst. no. **297** (1936)]. The author uses his approximate solution for finding critical loads, conditions and forms of unstable deformations, conditions at which the shell will remain permanently deformed. His parameters are: dimensions ratio and the greatest height of the middle surface (when the four corners of the shell rest on a plane) divided by the thickness. The second case when the supports at the boundaries of a loaded shell allow displacements is treated in a similar way, with a difference that the series used are not Fourier series. *T. Leser* (Aberdeen, Md.).

Buharinov, G. N. Axially symmetric deformation of a cylinder of finite length. Vestnik Leningrad. Univ. **11** (1956), no. 7, 77–86. (Russian)

If the stresses on the two ends of the cylinder are prescribed, the displacements can be expressed in the form of particular integrals together with series involving the Papkovich functions. These functions are the eigenfunctions of

$$\frac{d^4\phi}{dz^4}+2\lambda\frac{d^2\phi}{dz^2}+\lambda^2\phi=0,\ \phi=\frac{d\phi}{dz}=0 \text{ at } z=\pm h$$

and occur in the theory of plates [Papkovich, C. R. (Dokl.) Acad. Sci. URSS (N.S.) **27** (1940), 334–338; MR **2**, 332].

The difficulty is to determine the single set of coefficients in the series so as to satisfy two independent conditions on the curved boundary. Here the coefficients are expressed in terms of the solution of an integro-differential equation for the particular boundary conditions

$$\sigma_z=-q=\text{const at } z=h,\ \sigma_z=0 \text{ at } z=-h,\ \tau_{rz}=0 \text{ at } z=\pm h,$$

$$\sigma_r=0,\ \tau_{rz}=\frac{3}{8}\frac{r_0q}{h}(1-z^2/h^2) \text{ at } r=r_0.$$

R. C. T. Smith (Cambridge, Mass.).

Biot, M. A. Theory of deformation of a porous viscoelastic anisotropic solid. J. Appl. Phys. **27** (1956), 459–467.

The theory of deformation of porous, elastic, isotropic materials containing a viscous fluid was developed by the author in previous papers [cf. J. Appl. Mech. **23** (1956), 91–96; MR **17**, 804]. Now this work is extended to the case of porous, anisotropic, viscoelastic materials. The method of treatment is that of irreversible thermodynamics in the form developed by the author [J. Appl. Phys. **25** (1954), 1385–1391; **27** (1956), 240–253; MR **17**, 1035]. The general equations of the problem relating stress, deformation, fluid content, and fluid pressure, are established and discussed. Solutions are given for the isotropic case and for particular non-isotropic cases. As an illustrative example the settling of a loaded column supported laterally by a rigid impervious sheath is treated. *B. Gross* (Rio de Janeiro).

★**Cann, G. L. Non-linear waves in solids.** Proceedings of the Second Canadian Symposium on Aerodynamics, Toronto, 1954, pp. 238–263. The Institute of Aerophysics University of Toronto, Toronto, 1954.

The author uses a "plastic work hypothesis" to modify a stress-strain relation used by Malvern [J. Appl. Mech. **18** (1951), 203–208; MR **12**, 882] in work on longitudinal waves of plastic deformation in a bar of material exhibbiting a strain-rate effect. A numerical application of the modified relation predicts a region of constant strain at the impact, and of a specimen subjected to constant velocity impact. The Malvern relation does not predict such a plateau of uniform strain, which had been reported in experimental work [Duwez and Clark, Proc. Amer. Soc. Testing Materials **47** (1947), 502–532]. However, a recent, more detailed study [E. H. Lee, Div. Appl. Math. Brown Univ. Tech. Rep. no. **C11-5** (1955)] of the experimental results showed a marked deviation from the plateau of uniform strain and closer agreement with the results of Malvern. The re-assessment of the experimental results was unknown to the author at the time of writing but in addition the arguments advanced for the proposed modification appear unconvincing. *R. T. Shield.*

Petrašen', G. I. Propagation of elastic waves in stratified isotropic media separated by parallel planes. Leningrad Gos. Univ. Uč. Zap. **162.** Ser. Mat. Nauk **26** (1952), 191 pp. (Russian)

The problems are considered concerning the wave propagation in two- and n-layered halfspace, the physical interpretation of solutions and the field of displacements due to direct waves in particular, without quoting similar investigations outside USSR. From the latter the work differs essentially by the choice of stresses at the boundary which produce the disturbance. They are namely given by the formulas (in cylindrical coordinates)

$$T_z=\frac{q_0f(t)}{2\pi}\cdot\frac{n_0^2}{(1+n_0^2r^2)^{3/2}},\ T_r=-\frac{p_0f(t)}{4\pi}\frac{3n_0^4r}{(1+n_0^2r^2)^{5/2}},$$

where p_0 and q_0 are constants, $f(t)$ is a continuous differentiable function vanishing at the ends of a small interval $(0, T_0)$. It is approximated by the conditions: $f(t)=1$ for $0\leqq t\leqq T_0$ and vanishing for all other values of t. It is assumed that the parameter n_0 will approach ∞ at the end of the solution. The problem is split into two with $T_z^{(1)}=T_z$, $T_r^{(1)}=0$ and $T_z^{(2)}=0$, $T_r^{(2)}=T_r$. The potentials in the exact solution are taken in the form

$$\varphi_\nu=\int_0^\infty R_\nu(z,t,k)J_0(kr)dk,\ \psi_\nu=\cdots,$$

$$R_\nu=(2\pi i)^{-1}\int_{\sigma-i\infty}^{\sigma+i\infty}\chi_\nu(z,k,s)e^{st}ds$$

and as usual the series expansions are derived. The terms are interpreted as individual waves but the fact is stressed that some of the terms do not have a physical sense. A "principle of selection" is adopted stating that first the waves have to be considered which suffered one reflection at some interface (in exceptional cases — three reflections). Many terms are actually calculated. *W. Jardetzky* (New York, N.Y.).

Petrašen', G. I. Methods of investigation of wave-processes in media containing spherical or cylindrical boundaries of separation. Leningrad. Gos. Univ. Uč. Zap. 170. Ser. Mat. Nauk **27** (1953), 96–220. (Russian)

As in other papers of the author no results of similar investigations outside USSR are mentioned. In this paper, first, the solutions of two dimensional problems for a halfspace and a cylinder are compared. Use is made of Laplace transforms and Airy functions. Then, the

azimuthal waves in a circular cylinder and the phase functions are investigated. The calculation of the fast varying part of the field is given in a special section. Geometric optics of reflected waves, the "primary reflected wave" and multiply reflected waves are discussed in other sections. A section concerning some points of the Lamb's problem for an isotropic solid sphere concludes the investigation. *W. Jardetzky* (New York, N.Y.).

Eason, G.; Fulton, J.; and Sneddon, I. N. The generation of waves in an infinite elastic solid by variable body forces. Philos. Trans. Roy. Soc. London. Ser. A. **248** (1956), 575–607.

The variable body forces in question are usually dynamic point sources. Of particular interest and novelty are several treatments of moving point and disk sources. Some numerical results are given.

The authors reduce the problems to algebraic manipulations by using the Fourier transform. However they ignore the mathematical restrictions justifying this use. These restrictions are particularly important in elastic wave theory because the slow decay of the field quantities and their derivatives, both spacially and timewise, regularly leads to integrals that are not uniformly convergent. Indeed a number of integrals in this paper (c.f. above (4.1), above (4.2) when $k \to 0$, above (8.9), (11.4)–(11.6), above (12.4), (13.1)–(13.3), (14.7), (15.7)) diverge. While the authors' results are interesting and physically reasonable they must at present be regarded as formal and somewhat tentative. *E. Pinney.*

Iškov, P. K. On propagation of elastic waves in a stratum lying on an elastic base. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956,** 124–133. (Russian)

A posthumous paper in which the free waves in an elastic halfspace with a single layer were considered. To simplify the calculations the assumption that both media are incompressible has been made. The period equation in this twodimensional problem gives rise to two conditions. The first one gives the velocity of short Rayleigh waves at the free surface. Because of the interface they suffer a dispersion. The second equation is interpreted as giving a short wave which is produced by the wave along the interface. The dispersion in this case is due to the existence of the free surface. Tables and diagrams illustrate the conditions of the existence of the second wave. *W. Jardetzky* (New York, N.Y.).

Zvolinskiĭ, N. V.; and Skuridin, G. A. On an asymptotic method of solution of dynamical problems of the theory of elasticity. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956,** 134–143. (Russian)

Instead of deriving an approximation from an exact solution of a problem a method is developed to begin with an asymptotic solution for high frequencies, on assuming $\omega \to \infty$ in the boundary conditions. The method is shown in the case of reflection of a cylindrical wave from a plane boundary of an elastic halfspace. The Cauchy problem for the velocity potentials yields the result that the amplitudes of reflected waves can be expressed as products of two factors. One of these factors is termed a "reflection coefficient" the other is interpreted as a "divergence function". *W. Jardetzky.*

Garvin, W. W. Exact transient solution of the buried line source problem. Proc. Roy. Soc. London. Ser. A. **234** (1956), 528–541.

The problem considered is that of a line source in a semi-infinite elastic medium. The author finds that a special impulse-type source employed by Lapwood results in a solution in closed form. Laplace transform methods are used. A valuable feature of the author's work was an extensive numerical program. Graphs are given which show the development of the *P*-, *S*-, and Rayleigh pulses. *E. Pinney* (Berkeley, Calif.).

Martin, A. I. Some integrals relating to the vibration of a cantilever beam and approximation for the effect of taper on overtone frequencies. Aero. Quart. **7** (1956), 109–124.

The paper is in two parts. In Part I the author calculates certain integrals of the form $\int_0^l y^{(m)}y^{(n)}dx$, $\int_0^l xy^{(m)}y^{(n)}dx$, where $y(x)$ is the deflection of an untapered cantilever beam of length l under flexural vibration. In Part II these integrals are used to obtain estimates of the effect on the vibration frequency of uniform linear tapering in breadth and thickness. *R. N. Goss.*

Stabilini, Luigi. Problemi di instabilità elastica nelle costruzioni stradali e ferroviarie. Rend. Sem. Mat. Fis. Milano **25** (1953–54), 232–255 (1955).

See also: Gambill and Hale, p. 1086.

MATHEMATICAL PHYSICS

Optics, Electromagnetic Theory

Circuits

Ostrovskiĭ, G. M. On the construction of regions of stability. Avtomat. i Telemeh. **16** (1955), 501–507. (Russian)

The region of stability of a control system is the region in which the characteristic roots have negative real parts. Using Hurwitz's criterion, the author notes that the boundaries of the region of stability in the plane of a pair of coefficients of the characteristic polynomial are arcs of conic sections. These boundaries are studied for characteristic polynomials of fourth, fifth, and sixth degree.

J. P. LaSalle (Notre Dame, Ind.).

Krug, E. K.; and Minina, O. M. On peculiarities of investigation of dynamic properties of nonlinear systems having an unstable element. Avtomat. i Telemeh. **16** (1955), 536–541. (Russian)

A simple example of a nonlinear control system with an unstable element [a negative resistance] is used to show that the dynamic behaviour of the system cannot be predicted by the method of "harmonic balance". [This is a frequency-response method. The nonlinear element is assigned a transfer function on the basis of the fundamental in its response to a harmonic, and the frequencies at which the product of all the transfer functions of the system is unity are located.] The system is characterized by the equations

$$x_2 = \frac{k_1}{T_1p+1}x_1, \quad x_3 = f(x_2), \quad x_1 = \frac{k_2}{T_2p-1}x_3,$$

where k_1, k_2 are amplification factors and T_1, T_2 are time constants. The nonlinear function f is an odd function. In one case it is linear in an interval about the origin and constant outside this interval [it limits]; in the other case it is zero in an interval about the origin [a "deed space"], is then linear, and finally limits. A correct picture of the dynamic behaviour of the system is obtained by fitting the linear regions in the $(x_2, \dot{x}_2)$ phase plane.
J. P. LaSalle (Notre Dame, Ind.).

Strahov, V. N. Determination of certain fundamental parameters of magnetized bodies from data of magnetic observations. Izv. Akad. Nauk SSSR. Ser. Geofiz. **1956**, 144–156. (Russian)

A global interpretation method based on the use of integrals of suitably chosen functions — moments, for instance — of the observed and mapped quantity (simple integrals over a profile for a twodimensional problem, double integrals in a general case) is applied by the author to the interpretation of a magnetic profile L due to an infinite horizontal cylindrical magnetized body of a finite normal cross-section S, the profile L lying in the vertical plane of S.

Generalizing the reviewer's method [Geophysics **9** (1944), 463–493] and applying the theory of analytic functions, the author deduces reviewer's results concerning the determination of the inclination and the magnitude of the total magnetic moment of the body and the position of the center of gravity of magnetized masses, provided the magnetization is homogeneous and uniform. The mathematical treatment is very elegant.

It is hoped that this valuable article will attract the attention of specialists on the practical importance of this method. *E. Kogbetliantz* (New York, N.Y.).

Zmuda, Alfred J. Note on the components of magnetic intensity at inverse points relative to a spherical boundary. Trans. Amer. Geophys. Union **37** (1956), 273–274.

Galasiewicz, Z. On the collective motion in a system of particles having different masses and charges. Acta Phys. Polon. **14** (1955), 373–375.

Mit der Zerlegung der Bewegung eines Plasmas in einem sogenannten kollektiven und einem individuellen Anteil beschäftigten sich in den vergangenen Jahren besonders die Arbeiten von D. Bohm und D. Pines [Phys. Rev. (2) **82** (1951), 625–634; **85** (1952), 338–353; **92** (1953), 609–625; MR **12**, 886] sowie von J. Hubbard [Proc. Phys. Soc. Sect. A. **67** (1954), 1058–1068]. D. N. Zubarev [Dokl. Akad. Nauk SSSR (N.S.) **95** (1954), 757–760; MR **16**, 179] führte in die Wellenfunktion des Systems noch zusätzliche Veränderliche ein, welche die Fourierkomponenten des Dichteoperators sind. In der vorliegenden Arbeit verallgemeinert der Verfasser diese Methode für den Fall von verschiedenen Massen und Ladungen. (Elektronen und Ionen.) Mit Hilfe einer unitären Transformation der Wellengleichung erhält er dann leichter zu behandelnde Formeln, mit denen er als Beispiel die Bindungsenergien der Metalle Li, Na und K berechnet.
T. Neugebauer (Budapest).

Carini, Giovanni. Intorno al problema di una sfera conduttrice elettrizzata in moto. Ist. Lombardo Sci. Lett. Rend. Cl. Sci. Mat. Nat. (3) **19**(**88**) (1955), 631–638.

The problem of a conducting sphere moving in vacuum with uniform rectilinear velocity v is treated as a problem in relativistic electrodynamics. On the basis of the wave equation for a moving conductor, derived earlier by the author [same Rend. (3) **19**(**88**) (1955), 152–158; MR **17**, 561] and others, it is proved that the field inside the conductor is zero even with respect to the fixed frame of reference S. When terms in β^2 $(=v^2/c^2)$ are kept, it is shown that a surface charge distribution which is uniform relative to the moving frame of reference has relative to S its greatest density at the equator. *R. N. Goss.*

von Schelling, Hermann. Concept of distance in affine geometry and its applications in theories of vision. J. Opt. Soc. Amer. **46** (1956), 309–315.

With any two points in an n-dimensional affine space a specific primary (i.e. linear basis) determines $n+2$ simplices and the quotient of the volumes of two such simplices is an affine invariant. A symmetric function of these quotients is constructed that is a distance function (relative to the primary) over a subset of the affine space. It is shown that there is a projective transformation mapping this subset into a hyperbolic space such that the distance function becomes the hyperbolic distance.

When $n=1$ the subset of affine space is a half-line terminating in the primary c and the distance between x_1, x_2 is $\frac{1}{2}\ln[(x_2-c)/(x_1-c)]$. Application is suggested in psychophysics to the Weber-Fechner law for perceived brightness.

Two applications of the case $n=3$ to theories of visual perception are sketched. The first is concerned with the author's geometry for color space [same J. **45** (1955), 1072–1079]. The second concerns Luneburg's geometry of binocular vision. This application is questionable. As the author remarks: Luneburg's theory is highly controversial. Yet, this application is based, among other things, on Luneburg's postulate concerning the interchangeability of head and eye movements. This is not only a controversial postulate, it implies a contradiction in the theory. *G. L. Walker* (Southbridge, Mass.).

Tonolo, Angelo. Sur les intégrales des équations de l'électromagnétisme de M. L. de Broglie. C. R. Acad. Sci. Paris **242** (1956), 2626–2629.

De Broglie published his new theory of light [Une nouvelle théorie de la lumière, Hermann, Paris, 1940] in an attempt to establish a closer analogy between light and matter. In this theory, which of course contains the Maxwell theory as a special case, there are eight quantities describing the field. These quantities are related by fourteen partial differential equations, six of which connect the Maxwellian quantities, E, H, A, and V. The present article discusses the integration of these fourteen equations. The solutions are obtained by noting that the qunatities satisfy Klein-Gordon type equations, and using Weber's formula for the solution, which is then rewritten in an explicit form. The non-Maxwellian quantities are obtained by assuming that one of them, I_1, is zero. The boundary conditions are given by the knowledge of all these quantities at all points in space at zero time, and the quantities are obtained at all points in space for any positive time. *M. J. Moravcsik* (Upton, N.Y.).

Tonolo, Angelo. Sur les intégrales des équations de l'électromagnétisme de M. L. de Broglie. II. C. R. Acad. Sci. Paris **242** (1956), 2699–2702.

This paper treats the same problem as part I, but with different boundary conditions and only for the four Maxwellian quantities. Here we want to know the

quantities for any positive time in the interior of a volume S bounded by surface A, and we are given the quantities at zero time in S, and for all positive times on A. Just as the previous results, these also involve the Bessel function of first order in the integrand. *M. J. Moravcsik.*

Rogla, Vicente. A method of calculation for elastic links between plane structures. Las Ciencias 17 (1952), 571–636, 853–933 (1 plate). (Spanish)

This exposition was inspired by a very concrete problem: the study of the elastic links in a particular bridge being constructed in Barcelona. Remarking that matrices are singularly well adapted to the problem, the author begins with elementary notions and arrives ultimately at an exposition and elaboration of Fettis's paper on computing the coefficients of the characteristic equation [Quart. Appl. Math. 8 (1950), 206–212; MR 12, 209] with attention to repeated roots. Along the way he encounters and deals with a variety of special forms.

A. S. Householder (Oak Ridge, Tenn.).

Ferrari, Italo. Multipoli e onde di Schelkunoff dissimmetriche. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 18 (1955), 304–308.

Nachdem sich Verf. [dieselben Rend. (8) 17 (1954), 32–37; MR 17, 325] früher mit symmetrischen Shelkunoffschen Wellen befasst hat, wendet er sich jetzt den unsymmetrischen zu. Er zeigt, dass sich diese Wellen für den Fall $m=1$ von einer Kombination von Multipolen geeigneter Orientierung herleiten lassen, welche sich im Koordinatennullpunkt befinden. Beim Beweis geht er vom Felde eines Dipols aus und erhält die allgemeineren Feldausdrücke (er beschränkt sich auf das magnetische Feld) durch fortgesetzte Differentiationen, zuerst nach Cartesischen und dann nach Polarkoordinaten.

M. J. O. Strutt (Zürich).

Ferrari, Italo. Multipoli e onde di Schelkunoff. Il caso generale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 18 (1955), 623–630.

In der vorliegenden Arbeit verallgemeinert Verf. ein in der oben referierten Arbeit Ergebnis dahin, dass eine beliebige Shelkunoffsche elektromagnetische Welle durch eine Kombination von Multipolen erzeugt werden kann, welche mit geeigneter Orientierung und Stärke im Koordinatenursprung gewählt werden. Beim Beweis bringt er zunächst einige Umformungen bezüglich komplexer Differentiationen und geht dann für das magnetische Feld von einem Dipol aus. Durch geignete Differentiation des Dipolfeldes in Polarkoordinaten erhält er Ausdrücke, deren Summe mit geeignet gewählten Koeffizienten den allgemeinen Ausdruck für eine Shelkunoffsche elektromagnetische Welle ergibt. *M. J. O. Strutt.*

Aymerich, Giuseppe. Proprietà variazionali nelle guide d'onda anisotrope. Ann. Univ. Ferrara. Sez. VII. (N.S.) 4 (1954–1955), 9–25.

In a previous paper [Rend. Sem. Mat. Univ. Padova 22 (1953), 157–176; MR 15, 78] the author discussed wave propagation along the interior of a perfectly conducting anisotropic wave guide, which was the limiting form of a helical sheath built up out of a large number of perfectly conducting insulated wires each in the form of a helix; the current in such a sheath is always directed along the tangent to the helix.

He now goes on to consider an anisotropic coaxial cable, in which the inner conductor is of the form described above, but the outer is an ordinary perfect conductor. The problem of finding the modes of propagation along such a coaxial cable is reduced to a variational problem. It is shown that there is no critical frequency and that, in the case of a circular section, there are an infinite number of symmetrical modes. *E. T. Copson* (St. Andrews).

Aymerich, Giuseppe. Pseudoortogonalità dei modi e sviluppo formale di un campo armonico sostenuto da una guida anisotropa. Ann. Univ. Ferrara. Sez. VII. (N.S.) 4 (1954–1955), 27–32.

The author continues his study of the anisotropic coaxial cable described in the preceding review. He shows that the modes of propagation possess a property of pseudoorthogonality which can be used to derive a representation as a sum of normal modes of any harmonic electromagnetic field propagated along the cable.

E. T. Copson (St. Andrews).

Wait, James R. Low frequency radiation from a horizontal antenna over a spherical earth. Canad. J. Phys. 34 (1956), 586–595.

The problem of the horizontal electric dipole above a homogeneous sphere is formulated in the conventional fashion. The field is derived from two scalar Debye potentials, for which the author obtains expansions in the modes arising from separation of the wave equation in spherical coordinates. The individuality of the paper lies in its treatment of the boundary conditions; it is assumed that the tangetial electric and magnetic fields are related by a complex constant of proportionality, which for the E modes is set equal to the surface impedance for a horizontally polarized wave at grazing incidence, and for the H modes the same for a vertically polarized wave. Convergence of the resulting series for the radiated fields is improved by an application of the Watson transformation. The usual approximations are made for studying the distant field. For short distance the results agree with those of K. A. Norton [Proc. I. R. E. 25 (1937), 1203–1236].

R. N. Goss (San Diego, Calif.).

Kaufmann, A. Mise en équations et résolution des réseaux électriques en régime transitoire par la méthode tensorielle. Publ. Sci. Tech. Ministère de l'Air, Paris, Notes Tech. no. 56 (1955), v+104 pp. 1100 francs.

The purpose of this monograph is to give a rigorous treatment of general, linear, passive, networks (with lumped parameters) based on the concepts of Kron's tensor calculus. Emphasis is laid on the treatment of the transient behavior; therefore the original differential equations are submitted to a Laplace-Carson transform. The system is not supposed to be initially at rest; a distinction is made between forced initial conditions due to external forcing and hereditary conditions which are remnants of the state that preceded the application of external forces. The treatment, which is systematic, is given in four main chapters: Derivation of Kirchhoff's laws; general equations for loop currents; general equations for the potential differences between branch points; complex systems. *B. Gross.*

Duffin, R. J.; and Keitzer, Elsa. Formulae relating some equivalent networks. J. Math. Phys. 35 (1956), 72–82.

The Brune method of network synthesis makes use of ideal transformers. It has been shown by R. Bott and R. J. Duffin [J. Appl. Phys. 20 (1949), 816; MR 12, 307] that for an arbitrary, passive, linear, network there exists

always a network without transformers having the same driving point impedance. The present paper gives explicit formulae and a discussion of the particular case of the general two-loop passive network. The result is achieved in two steps, first a reduction to a network with real transformers and then a reduction to a network without transformers. Explicit circuit diagram are given.

B. Gross (Rio de Janeiro).

Brachman, Malcolm K.; and MacDonald, J. Ross. Generralized immittance kernels and the Kronig-Kramers relations. Physica **22** (1956), 141–148.

It is shown that the solution of a nth-oder linear differential equation with constant coefficients and forcing function exp $(i\omega t)$ satisfies the Kronig-Kramers relations [Kramers, Atti Congresso Internaz. Fis., 1927; Como-Pavia-Roma, v. 2, Zanichelli, Bologna, 1928, pp. 545–557; Kronig, J. Opt. Soc. Amer., **12** (1926), 547–557]. The result is extended to integrals over these solutions.

B. Gross (Rio de Janeiro).

★**Laurent, Torbern. Vierpoltheorie und Frequenztransformation. Mathematische Hilfsmittel für systematische Berechnungen und theoretische Untersuchungen elektrischer Übertragungskreise. Aus dem Schwedischen übersetzt von N. v. Korshenewsky.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1956. xii+299 pp. DM 34.50.

★**Hofmann, O. Beitrag zur Theorie des anallaktischen Punktes entfernungsmessender Fernrohre mit negativer Zwischenlinse.** Jenaer Jahrbuch **1955**, 1. Teil, pp. 15–27. Gustav Fischer Verlag, Jena, 1955.

A determination of the error in stadia surveying with an internal focussing theodolite produced by some manufacturing tolerance and adjustment errors in the instrument. Errors in focal lengths and cross-hair separation are considered but decentering and other errors are not. The treatment is simpler and more comprehensive than a previous study by H. Köhler [Jenaer Zeiss-Jahrbuch **1950**, pp. 101–124; Z. Vermessungswesen, **76** (1951), 65–73].

G. L. Walker (Southbridge, Mass.).

Oterma, L. Recherches portant sur des télescopes pourvus d'une lame correctrice. Ann. Univ. Turku. Ser. A. **19** (1955), 1–134.

"A l'origine le présent ouvrage avait pour but de présenter les méthodes et les formules qui ont été utilisées à Turku pour les calculs de construction des télescopes anastigmatiques, mais au cours du travail cette tâche s'est étendue. On a réussi à améliorer quelque peu les méthodes et les formules et de nouvelles questions se sont présentées. Comme il n'existe pas un exposé uniforme sur ce sujet, on a voulu composer une sorte de manual qui pourrait être utilisé pour des recherches de ce genre." (Résumé de l'auteur.)

An invaluable reference on the design of Schmidt-type systems.

G. L. Walker (Southbridge, Mass.).

Teodorescu, N. Les fondements de la théorie invariante de la propagation des ondes. I. Analyse des principes de Huygens. An. Acad. R. P. Române. Ser. Mat. Fiz. Chim. **3** (1950), 611–630. (Romanian. Russian and French summaries)

There are two theories of wave-propagation, viz. (i) the original geometrico-kinematical method of Huygens, which regards the wave-front at any instant as the envelope of secondary waves emitted from all the points of the wave-front at some earlier instant and (ii) the physical method which discusses the variation of some physical-quantity (such as the pressure in acoustics or the electromagnetic field in optics) in the propagation of the wave.

The author attempts to unify the two points of view by regarding each element of a wave surface as a contact-element which he calls a corpuscle.

This is merely the first non-mathematical part of the memoir, and we must await the sequel to see what advantages this theory may have.

E. T. Copson.

See also: Rosenbloom, p. 1100; Pham Mau Quan, p. 1144; Weinstein, p. 1091.

Quantum Mechanics

Godnev, I. On the foundation of classical quantum statistics. Ivanov. Gos. Ped. Inst. Uč. Zap. Fiz.-Mat. Fak. **1** (1941), no. 1, 42–46. (Russian)

Kar, K. C.; and Mukherjee, H. Classical derivation of the pseudoscalar interaction potential. Indian J. Theoret. Phys. **1** (1953), 67–72.

The difference between two forms of nucleonic interaction given by Pauli [Meson theory of nuclear forces, Interscience, New York, 1946; MR **8**, 428] and Bethe [Phys. Rev. (2) **57** (1940) 260–272] is explained by whether one takes pole strength or magnetic moment as the basic idea in the classical analogy.

A. J. Coleman.

★**Février, Paulette. L'interprétation physique de la mécanique ondulatoire et des théories quantiques.** Gauthier-Villars, Paris, 1956. viii+216 pp. 3200 francs.

This is a careful, detailed, and complete study of the various philosophical questions which arise in quantum mechanics, the most important question being that of causality versus indeterminism. In spite of recent doubts raised by L. de Broglie [La physique quantique, restera-t-elle indéterministe, Gauthier-Villars, Paris, 1953; MR **16**, 1183], the author's general position is that any thoroughgoing determinism remains inconsistent with quantum mechanics, and that J. von Neumann's proof of this fact retains its validity.

The chapter headings are: I. Adéquation d'une théorie physique. II. Théorie générale des prévisions. III. La décomposition spectrale. IV. Caractère ouvert d'une Mécanique ondulatoire. V. Mesures et schémas objectivistes. VI. Interprétation prévisionnelle et interprétation réaliste. VII. Microphysique et cosmologie. VIII. La représentation fonctionnalle. IX. Considérations de structure logique. X. La methode du fluide associé de M. Destouches.

The pilot-wave and double solution theories of de Broglie are carefully examined, as are the hydrodynamic quantum theory and the theory of previsions of J. L. Destouches. The chapter on logic considers the n-valued and modal logics as well as the quantum logic of Birkhoff and von Neumann. There is a good bibliography, index, and table of symbols.

O. Frink.

Kar, K. C.; and Sanatani, S. The classical interpretation of Dirac's theory of electron. Indian J. Theoret. Phys. **1** (1953), 1–24.

The announced purpose of the present paper is to obtain the same results as Dirac's theory of the electron

without the use of matrices or non-commuting symbols. The desirability of this is not evident to the reviewer and in any case the present attempt is vitiated by a number of errors. To describe the authors' equation (15) as "without spin" is to miss the point of Dirac's theory. The transition from equation (22) to (23) is a non-sequitur. To claim, as the authors do in their last sentence, that Dirac obtains only one of their thirty-two sets of equations is to fail to observe that Dirac's results are independent of the choice of sign of his five roots of unity, i, α_μ. *A. J. Coleman.*

Sanatani, S. Relativistic Hamiltonian and wave equations of an electron in an assembly. Indian J. Theoret. Phys. 2 (1954), 79–86.

An attempt is made to overcome the unsatisfactory manner in which the spin orbit interaction was introduced in the paper reviewed above. Unhappily, the paper is based on the following elementary error, previously implicit, now explicit: "it seems quite reasonable to assume from physical considerations that if the averages of two quantities belonging to the phase space of an assembly are equal, the two quantities may be taken equal". This is equivalent to asserting that if in a set of mixtures of oranges and apples, the average number of apples and oranges are equal, then an apple is an orange.

A. J. Coleman (Toronto, Ont.).

Park, D. The theorem on incoming waves. Nuovo Cimento (10) 3 (1956), 979–987.

The use of ingoing instead of outgoing waves in certain problems of scattering theory involving two interactions has in the past been justified by proofs for special cases and plausibility arguments in general [cf. Breit and Bethe, Phys. Rev. (2) 93 (1954), 888–890, and their references to earlier work; MR 15, 919]. The author succeeds in eliminating the remaining foggy notions surrounding this point by a simple and rigorous proof of the following general theorem. Let $i\hbar\partial_t\psi=H\psi$, $i\hbar\partial_t\varphi=H_0\varphi$, and $H=H_0+V_1+V_2$. Then the transition amplitude from the state φ to the state φ', $T=(\varphi', (V_1+V_2)\psi^{(+)})$, can be written

$$T=(\varphi', V_1\psi_1^{(+)})+(\psi_1'^{(-)}, V_2\psi^{(+)}),$$

where

$$\psi_1^{(+)}=\varphi+\lim_{\gamma\to+0}(E-H_0-V_1+i\gamma)^{-1}V_1\varphi,$$
$$\psi^{(+)}=\varphi+\lim_{\gamma\to+0}(E-H+i\gamma)^{-1}(V_1+V_2)\varphi,$$
$$\psi_1'^{(-)}=\varphi'+\lim_{\gamma\to+0}(E-H_0-V_1-i\gamma)V_1\varphi'.$$

The paper also clarifies a number of related questions in bremsstrahlung, perturbation theory, and field theory.

F. Rohrlich (Iowa City, Ia.).

Landé, Alfred. Quantentheorie auf nicht-quantenhafter Grundlage. Naturwissenschaften 43 (1956), 217–221.

Fantappiè, Luigi. Costruzione di un sistema fondamentale di operatori fisici differenziali, per ogni universo a gruppo base semplice. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 19 (1955), 213–217.

Fantappiè, Luigi. Deduzione autonoma dell'equazione generalizzata di Schrödinger, nella teoria di relatività finale. Atti Accad. Naz. Lincei. Rend. Cl. Sci. Fis. Mat. Nat. (8) 19 (1955), 367–373 (1956).

These papers deduce the Klein-Gordon equation in quantum mechanics as a limit, when the radius of the universe becomes infinite, of a classical (non-quantized) equation in the author's extension of relativity, which is based on a simple (pseudo-orthogonal) group having the Lorentz group as a type of limit. *I. E. Segal.*

Gellikman, B. T. On the theory of strong coupling for meson fields. I, II. Ž. Eksper. Teoret. Fiz. 29 (1955), 417–429, 430–438. (Russian)

These are two of a series of three papers (see following review for III) on strong coupling perturbation theory methods applied to scalar and pseudoscalar mesons interacting with nucleons. A part of the results were reported briefly in Dokl. Akad. Nauk SSSR (N.S.) 90 (1953), 359–362, 991–994; 91 (1953), 39–42, 225–228 [MR 15, 379, 918; 16, 918]. The nucleons are treated as infinitely heavy extended sources and no creation of nucleon-anti-nucleon pairs is allowed. A variety of couplings is considered: scalar and pseudo vector, neutral and charged.

The author makes an analogy between two methods of treating the motion of the nuclei in a molecule and two methods in strong coupling theory for mesons. In the first method, the unperturbed hamiltonian of a molecule is that of the electrons with the nuclei held fixed (kinetic energy of nuclei omitted.) This makes the zeroth approximation wave functions and energies dependent on the coordinates of the nuclei. The perturbation is then the kinetic energy of the nuclei. In the second method, the unperturbed hamiltonian is the same as before except that the coordinates of the nuclei are given some fixed values to be chosen afterwards. The zeroth approximation wave functions and energies then depend on these parameters but not on the actual position of the nuclei. The perturbation is then the kinetic energy of the nuclei plus the difference of the actual potential energy and the potential energy with the nuclei in the previously mentioned fixed positions. Similarly, in the case of mesons, the zeroth approximation wave functions can be taken as depending on the actual values of the meson field strength and the perturbation as the kinetic energy alone (first method). Alternatively, one can take the meson field strength as arbitrary but fixed in the zeroth approximation wave functions and energies and use as perturbation the total hamiltonian minus the "potential energy" of the fixed distribution of meson field strength. The author uses mainly the second method, asserting that it is very much simpler than the first, at least as long as his zeroth approximation solutions turn out to be non-degenerate. Giving numerical details only for purposes of illustration he indicates the chain of calculation leading to the following results. In zeroth approximation (order g^2) the meson field is classical, does not describe actual mesons but accounts for nucleon self energy. In first approximation (order g^0), the meson field becomes an operator describing physical mesons which do not interact with each other but may be bound to nucleons (isobars). In second approximations (order g^{-2}), the true mesons scatter and produce corrections to the nucleon self energy. Expressions are derived for nucleon-nucleon forces, the magnetic moment of the nucleon, meson nucleon scattering and isobar energies. There are significant differences between the results and those of previous workers on strong coupling theory. The author attributes the discrepancies to essential errors in previous calculations.

A. S. Wightman (Princeton, N.J.).

Gelikman, B. T. On the theory of strong coupling for meson fields. III. Ž. Eksper. Teoret. Fiz. **29** (1955), 572–584. (Russian).

In this paper (see the preceding review for its predecessors), the author continues the exploration of his own version of strong coupling theory, considering a variety of problems and giving a detailed outline of the calculations of the wave functions and energy values up to the point where numerical work or renormalization would be necessary to go further. He begins with the problem of a pseudo scalar meson field coupled to n nucleons described by the Dirac equation. Here recoil is taken into account but not nucleon anti-nucleon pair production. The coupling is always taken as smeared out in space but the transition to zero source radius is discussed. At one point or another the following couplings are considered: neutral and charge symmetrical pseudo scalar and pseudo vector, and charge symmetrical scalar. The last half of the paper is devoted to the problem of nucleon and pseudo scalar meson fields coupled with a charge symmetrical coupling, which requires a generalizations of the previously used strong coupling theory. An approximate expression for the Green's function of a nucleon is calculated. *A. S. Wightman.*

Salam, A. On generalised dispersion relations. Nuovo Cimento (10) **3** (1956), 424–429.

A dispersion relation is derived which connects the real and imaginary parts (dispersive and absorptive parts) of the meson-nucleon scattering amplitude for non-zero angles. The equation relates amplitude parts which do not refer to the same angle in the center of mass system. The result is a generalization of the Kramers-Kronig dispersion relation for light to non-vanishing angles [cf. Bethe and the reviewer, Phys. Rev. (2) **86** (1952), 10–16] and to non-vanishing masses. The derivation makes use of micro-causality (the fields commute outside the light cone) and of a special Lorentz frame which simplifies the argument considerably. As in a paper on the same subject by Goldberger [ibid. **99** (1955), 979–985] the derivation requires the interchange of two integrations which is valid only under certain assumptions about regularity and asymptotic behavior of the amplitude. It is not known whether these conditions are actually met. A major difficulty in this connection is the evaluation of the contributions from bound states. *F. Rohrlich.*

Salam, Abdus; and Gilbert, W. On generalized dispersion relations. II. Nuovo Cimento (10) 3 (1956), 607–611.

The paper reviewed above is generalized to the case where the scattering meson can cause a spin flip of the nucleon. Only neutral mesons are considered in detail. The special Lorentz frame and the assumptions previously made are again necessary. The results are the same as those obtained independently by Oehme [Phys. Rev. (2) **100** (1955), 1503–1512] and others. The paper concludes with a remark concerning the non-unique definition of the renormalized coupling constant which enters the bound state contribution to the absorptive parts of the amplitude. *F. Rohrlich* (Iowa City, Ia.).

Nigam, B. P.; and Foldy, L. L. Representation of charge conjugation for Dirac fields. Phys. Rev. (2) **102** (1956), 1410–1412.

The charge conjugation for fermion fields is given explicitly in forms which make no explicit reference to the expansion of the field into orthonormal functions. Depending on the choice of an auxiliary operator the actual transformation is given in several explicit forms. The equivalence of these various forms is shown only for the case when the Wigner-Jordan anticommutation rules for the field operators are given in an irreducible representation. One of the forms of the transformation operator agrees with Schwinger's C operator [Phys. Rev. (2) **74** (1948), 1439–1461; MR **10**, 345]. The general form of the charge conjugation transformation operator is then used to resolve an apparent paradox according to which the commutation relations between transformation operators for charge conjugation and space inversion would give information about a phase factor which is supposed to be arbitrary. It is shown that if the total charge is zero, the two operators commute independently of the value of this phase factor. On the other hand, if the total charge is not zero, the charge conjugation operator is not an observable since it does not commute with the total charge, and hence the phase factor is again unobservable. *M. J. Moravcsik* (Upton, N.Y.).

Utiyama, Ryoyu. Invariant theoretical interpretation of interaction. Phys. Rev. (2) **101** (1956), 1597–1607.

The general problem discussed here is as follows. Let us consider a set of fields, Q^A $(A=1, 2, \cdots, N)$, with a Lagrangian $L(Q^A, Q_{,\mu}{}^A)$, $Q_{,\mu}{}^A=\partial Q^A/\partial x^\mu$ and let us assume that the Q^A's are invariant under a transformation group G which depends on parameters $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$. Now let us replace G by a wider group G' which depends instead of on the ε's on a set of arbitrary functions $\varepsilon_1(x), \varepsilon_2(x), \cdots, \varepsilon_n(x)$. This replacement is equivalent to the introduction of another field A'^J $(J=1, 2, \cdots, M)$, which interacts with the Q^A's. One can then ask the following questions. 1) What kind of A'^J is introduced by the change to G'? 2) How does A'^J transform under G'? 3) What is the form of the interaction between A'^J and Q^A? 4) What is the new Lagrangian $L'(Q^A, Q_{,\mu}{}^A, A'^J)$? 5) What are the field equations for A'^J? These questions are answered in full generality. Thus, for instance, it is shown that

$$L'(Q^A, Q_{,\mu}{}^A, A^a{}_\mu)=L''(Q^A, \nabla_\mu Q^A),$$

where $A^a{}_\mu=C^{-1a}{}_{\mu,J}A'^J$ and

$$\nabla_\mu Q^A=\partial Q^A/\partial x^\mu-T_{(a),B}{}^A Q^B A^a{}_\mu,$$

where T and C are matrices determining the variation of Q^A and A'^J respectively. The general discussion is followed by a treatment of special examples. The first of these is the interaction of a charged field Q, Q^* with the electromagnetic field A_μ. Here G is the group of phase transformations, G' the group of gauge transformations, and we have the familiar $\nabla_\mu Q^A=\partial Q^A/\partial x^\mu-iA_\mu Q^A$. The second example is the interaction of the nucleon field with the Yang-Mills field [Phys. Rev. (2) **96** (1954), 191–195; MR **16**, 432]. In this case G is the rotation group in isotopic spin space, and G' is obtained by considering $\delta\psi^\alpha= i\sum_{c=1}^3 \varepsilon^c\tau_{(c)\beta}{}^\alpha\psi^\beta$ (ψ^α is the nucleon wave function in isotopic spin space, and the $\tau_{(c)}$'s are the isotopic spin matrices) and changing the ε^c's to $\varepsilon^c(x)$'s. Finally the case of the gravitational field is considered, where G is the Lorentz group, G' is a generalization of the Lorentz group analogous to the case of the Yang-Mills field, and the resultant new field is essentially the Christoffel affinity $\Gamma_{\mu\nu}{}^\lambda$. *M. J. Moravcsik* (Upton, N.Y.).

Caldirola, P.; e Duimio, F. Introduzione di una lunghezza fondamentale nella teoria classica dell'elettrone. Nuovo Cimento (9) **12** (1954), 699–732.

The Lorentz equation for the force on an electron is

replaced by the following difference equation:

$$-\frac{m_0}{\tau_0}\left[u_\alpha(\tau-\tau_0)+\frac{u_\alpha(\tau)u_\beta(\tau)}{c^2}u_\beta(\tau-\tau_0)\right]=e/c\, F_{\alpha\beta}u_\beta(\tau),$$

where τ is intrinsic time and $\tau_0=4e^2/3m_0c^3$ is a new universal constant corresponding to a fundamental length $\tau_0 c$. The particular choice of τ_0 ensures that, when the left-hand side of the above equation is expanded in a Taylor's series, the familiar expression for radiation force appears. The chief advantage of the difference equation is that it avoids the initial condition difficulty of theories such as those of Lorentz, Dirac or Eliezer in which the motion of a charge is determined by a third- or higher order differential equation. The difference equation is solved for five particular simple types of motion. The Lagrangian formulation of the theory is complicated so that its quantization would probably be extremely difficult. The non-relativistic approximation to the above difference equation is shown to coincide with the differential equation of infinite order proposed by Page [Phys. Rev. (2) **11** (1918), 376–400]. It is shown that the theory can be inserted into the frame of a general theory of the electromagnetic field interacting with its own sources in which the mass of the electron is completely e.-m. in character, without violating the principle of relativity. Making use of periodic solutions of the equations $u_\alpha(\tau)-u_\alpha(\tau-\tau_0)=0$ an internal motion is ascribed to the electron which may provide an explanation of the anomolous magnetic moment. *A. J. Coleman.*

Lozano, Juan Manuel. Dynamical description of the scattering by a potential. Rev. Mexicana Fis. **2** (1953), 155–179. (Spanish. English summary)

A Green's function is obtained explicitly in terms of Bessel functions and the S matrix which provides a description of the time development of a non-relativisitic scalar wave packet scattered by a static short-range central potential. Restrictions on the position of the poles of the S matrix are obtained. *A. J. Coleman.*

Finkelstein, R. J.; and Moe, M. Scattering by a symmetric potential. Phys. Rev. (2) **100** (1955), 1775–1779.

The scattering by a symmetric structure, such as a regular molecule or a crystal, is studied by labelling phase-shifts according to the irreducible representations of the crystallographic groups. The determination of the phase-shifts is performed by means of Schwinger's variational principle, in which trial functions of the appropriate symmetry are introduced. These are selected by means of projection operators. The scattering of electrons by a tetrahedral molecule is treated as an example. *M. Cini* (Turin).

Lozano, Juan Manuel. Dynamical description of scattering by means of tensor forces. Rev. Mexicana Fis. **4** (1955), 13–22. (Spanish. English summary)

The work reviewed above is extended to scattering by tensor forces, providing an example of a two channel nuclear reaction for which the S matrix is a meromorphic function of the wave number. *A. J. Coleman.*

Sen, P. A mass spectrum from a field theory model of the non-local theory. Nuovo Cimento (10) **3** (1956), 612–625.

The author writes the wave function of an elementary particle as a product of one function involving only the "external" coordinates of the particle and another function depending on only the "internal" coordinates. [Cf. H. Yukawa, Phys. Rev. (2) **77** (1950), 219–226; **80** (1950), 1047–1052; MR **11**, 567; **12**, 571.] The external factor of the wave function is supposed to obey the usual differential equations of quantized field theories while a new equation is assumed for the other factor. The external and internal degrees of freedom are thus supposed to be entirely decoupled. By choosing a particular form for the wave equation for the internal motion (the reason for this choice is not clear to the reviewer) the author obtains a mass spectrum that shows some qualitative similarity to the known masses of the elementary particles although more masses are obtained than those which are experimentally known today. *G. Källén* (Copenhagen).

Sen, P. A simple non-local quantum electrodynamics. Nuovo Cimento (10) **3** (1956), 390–408.

Following an idea by Wataghin [Z. Phys. **88** (1934), 92–98] the author modifies the Dirac-Maxwell equations by replacing the local interaction term $ie\gamma A(x)\psi(x)$ by $ie\gamma(A(x-b_e)+A(x+b_e))\psi(x)$. Here, b_e is a four vector that is a measure of the non-local character of the interaction. With this basic assumption, second order calculations are carried through for various quantities and it is shown that finite results are obtained. Higher orders are not discussed. In this connection, the reviewer would like to draw the attention to the work by Stückelberg and Wanders [Helv. Phys. Acta **27** (1954), 667–682; MR **16**, 1185] and by P. Kristensen [cf. C. Møller, Proc. Internat. Conference Theoret. Phys., Kyoto and Tokyo, 1953, Science Council of Japan, Tokyo, 1954, pp. 13–23; MR **16**, 656]. For a very general class of non-local theories (including the one discussed here) these authors have shown that the convergence factor will always destroy some general properties of the theory like the unitarity of the S-matrix or macroscopic causality of the interaction if the details are worked out in higher orders. *G. Källén.*

Roos, Oldwig v. Ein neues Verfahren zur Beseitigung der Selbstenergiedivergenzen in der Quantenelektrodynamik. Z. Physik **144** (1956), 323–335.

The author introduces an auxiliary variable l and a "form factor" $f(l)$ in the conventional expansion of the radiation field $\psi_\nu(x)$ in plane waves and writes

$$\psi_\nu(x)=\sqrt{\left(\frac{\hbar c}{(2\pi)^4}\right)}\int dl\int d^3k\sqrt{\left(\frac{f(l)}{2|\bar{k}|}\right)}\{a_\nu(l,\bar{k})e^{i(1+l)kx}$$

$$+a_\nu(l,k)^+e^{-i(1+l)kx}\}.$$

If the function $f(l)$ is put equal to $2\pi\cdot\delta(l)$ this expansion is identical with the usual one. The author then remarks that the delta-function can be represented with the aid of a complex integral in the following way

$$\int_{-\infty}^{+\infty} g(l)\delta(l)dl=(2\pi i)^{-1}\oint\frac{g(l)}{l}dl.$$

For this relation to hold it is necessary that the function $g(l)$ can be extended to an analytic function of l regular in some domain around the origin. The author then writes the self energy terms in second order perturbation theory in such a form that they contain either $\int f(l)dl/l(l+1)$ or $\int f(l)dl/l^2$ as a factor. If these integrals are evaluated directly with $f(l)=2\pi\cdot\delta(l)$ they are both infinite. This corresponds to the well-known divergences in second order perturbation theory. The author then suggests that one should use the above complex integral representation

of the delta-function in spite of the fact that the functions appearing in these applications do not fulfill the necessary regularity conditions. By choosing $f(l)=2\pi(1+l)\cdot\delta(l)$ he obtains the value zero for both these integrals and in this way the divergent self energy in second order perturbation theory has been "eliminated". *G. Källén.*

Dyson, Freeman J. General theory of spin-wave interactions. Phys. Rev. (2) **102** (1956), 1217–1230.

This is the first of two important papers discussing the spin-wave theory of the idealized Heisenberg Hamiltonian for a ferromagnet. From a physical point of view these papers dispose of doubts raised by a number of authors as to the validity of the spin-wave concept even in this idealized case, and go a long way towards clarifying the question of how spin-waves interact and what their relationship is to the high-temperature approximations. However, the methodology and its possible further application to this and other problems may be of even greater interest.

The merit of Dyson's technique rests on the following idea: that this problem contains two separate types of non-linearities — i.e., interactions containing the spin amplitudes in more than second order which cannot be diagonalized out by use of symmetry: first, non-linearities of the real, physical Hamiltonian, which he calls "dynamical" interactions; and second, non-linearities caused by the constraint that the magnitudes of the separate spins are constants, called "kinematical." The dynamical interactions are minimized by the natural, physical choice of basis functions, but because of the constraint this basis is not an orthonormal one. The methodological interest in the papers lies in the techniques for dealing with the problem of the kinematical interaction in this form and showing that its effect can be neglected.

This is done by constructing an "ideal" model in which the constraints are eliminated and the basis orthonormal, and studying the metric and projective operators necessary to calculate physical quantities from results in the ideal model.

The second half of this paper is devoted to calculating eigenstates, scattering matrices, and Green's functions in the ideal model. The final two sections express the exact partition function in terms of the quantities calculable with this technique; the next paper will evaluate this result. *P. W. Anderson* (Murray Hill, N.J.).

Dyson, Freeman J. Thermodynamic behavior of an ideal ferromagnet. Phys. Rev. (2) **102** (1956), 1230–1244.

In this, the second paper of two, Dyson evaluates the first few terms of the power-series in T of the partition function of the Heisenberg ferromagnet, proving in particular that the leading correction terms are those coming from more exact evaluation of the linear theory, that smaller terms come from the dynamical interaction, and that the kinematical interaction caused by the constraints does not enter the power series at all, giving only $e^{-a/T}$ terms. To do this he further explains the relationship of the physical problem to the ideal model and modifies his "ideal model" in such a way that all states not physically realized are a rather large, finite distance above the lowest energy. Clearly then the corrections to it only enter in exponential terms. After this is proved, the major part of the paper is devoted to actual calculation of this virial series for the free energy. *P. W. Anderson.*

Umezawa, H.; Tomozawa, Y.; Konuma, M.; and Kamefuchi, S. High energy behaviour of renormalizable fields. Nuovo Cimento (10) **3** (1956), 772–799.

This chapter contains a review of the work of Gell-Mann and Low [Phys. Rev. (2) **95** (1954), 1300–1312; MR **16**, 315], Lehmann [Nuovo Cimento (9) **11** (1954), 342–357; MR **17**, 332], Lee [Phys. Rev. **95** (1954), 1329–1334; MR **16**, 317] and others on the mathematical consistency of the renormalization procedure in quantum field theory. The authors discuss the relationship of the intrinsic masses and coupling constants with the observed masses and coupling constants, and it is suggested that the renormalization theory may give unreliable results for high-energy processes even in the case of renormalizable interactions. *S. N. Gupta* (Lafayette, Ind.).

Liotta, R. S. Covariant canonical equations for a classical field. I. Nuovo Cimento (10) **3** (1956), 438–446.

In this paper it is suggested that in the classical field theory it may be useful to define the Hamiltonian as the trace of the canonical energy-momentum tensor. Using this new invariant definition of the Hamiltonian, the author derives the Hamilton equations for a system of fields from the Lagrangian density along the usual lines. He also derives the Hamilton-Jacobi equations, and discusses the relation between the Jacobi functions and the canonical transformations. *S. N. Gupta.*

Mossin Kotin, Cecilia. Localization of a quantized field and its fluctuations. An. Acad. Brasil. Ci. **27** (1955), 395–403.

It is shown that to a quantized radiation field can be associated a classical one, the amplitudes of the latter coinciding with the mean values of the former. Since a radiation field may be considered as a superposition of harmonic oscillators this result just represents the correspondence, in the oscillator problem, between the motion of appropriate wave packets and classical solutions. As in that case, this equivalence holds only after the quantum zero point energy has been subtracted, and entails the indeterminacy of the number of photons of the quantum field. For a representation diagonal in the number of photons, an associated classical field is introduced which describes their propagation and interaction, but only in an approximate fashion. Finally, the energy density fluctuations of the free radiation field are studied, and their mean square deviation in vacuum and one photon states derived. *S. Deser* (Copenhagen).

Corinaldesi, E. The two-body problem in the theory of the quantized gravitational field. Proc. Phys. Soc. Sect. A. **69** (1956), 189–195.

A general review is given of previous papers by the author and collaborators on a modification of classical electron theory. The present attempt introduces a fundamental proper time interval such that dynamical variables are linked within it, and relativistically invariant difference equations of motion are then given for a classical particle, which reduce to the usual ones as this interval goes to zero. Arguments from the uncertainly principle are adduced to connect this time to the particle's rest mass and possible spectrum of internal excited states. The "runaway solutions" and apparent lack of causality of previous attempts are here avoided, on the macroscopic average. Applications to some particular motions are given and it is shown that a physically sensible behavior ensues. It is also found that the singularity of the coulomb

potential disappears near the origin. Periodic internal solutions can be superposed on the macroscopic ones in such a way that an anomalous magnetic moment can be ascribed classically to such internal motion. In view of the necessity of quantizing the classical theory, the difficulties of quantizing via a Hamiltonian method for finite differences (or equivalently, infinitely many time derivatives) are considered. As has been pointed out by Feynman, his lagrangian quantization by path integration is also inapplicable to such a problem. *S. Deser.*

Kastler, Daniel. Algèbre multilinéaire et quantification du champ des photons. C. R. Acad. Sci. Paris **242** (1956), 2445–2448.

A treatment of the quantized photon field along the lines due to J. M. Cook [Trans. Amer. Math. Soc. **74** (1953), 222–245; MR **14**, 825] in terms of skew-symmetric tensors. *I. E. Segal* (Chicago, Ill.).

Kahana, S.; and Coish, H. R. Classical meson theory. II. Amer. J. Phys. **24** (1956), 390–399.

d'Espagnat, B.; and Prentki, J. Symmetries in isotopic spin space and the charge operator. Phys. Rev. (2) **102** (1956), 1684–1685.

Fulton, T.; and Newton, R. G. Explicit non-central potentials and wave functions for given S-matrices. Nuovo Cimento (10) 3 (1956), 677–717.

The construction of potentials from phase shifts and bound state energies for central forces by Bargmann, Gel'fand and Levitan, and Jost and Kohn, is here generalized to non-central potentials. A 2×2 sub-matrix $S(k)$ of the S-matrix, associated with angular momenta l and $l+2$ is considered. It was previously proven for S-states by Newton and Jost [Nuovo Cimento (10) **1** (1955), 590–622; MR **17**, 155] and generalized to any l by Newton [Phys. Rev. (2) **100** (1955), 412–428] that a short range potential (second absolute moment exists), if it exists, is uniquely determined by $S(k)$, the energies of the L bound states associated with l and $l+2$, and L real, symmetric, positive semi-definite matrices. In the present paper the experimental S-matrix elements are assumed to be approximated by rational functions of $k=\sqrt{E}$. The determination of $F(k)$ where $S(k)=F(k)F^{-1}(-k)$ can then be accomplished by a purely algebraic procedure and there is no need for the general procedure based on a theorem by Plemelj. From $F(k)$ the spectral function can be constructed which determines the Gel'fand-Levitan equation. In the case of no bound state the author is able to solve this equation under the assumption that (^{T} means "transposed")

$$T(k^2)[F^T(k)F(-k)]^{-1}=T(k^2)/\prod_{i=1}^{\nu_\alpha}(k^2+\alpha_i^2) \quad (\mathrm{Re}\,\alpha_i>0;\ i=1,\cdots,\nu_\alpha),$$

where $T(k^2)$ is a symmetric matrix whose elements are polynomials in k^2 and det $T(k^2)$ has simple zeros only, at points $k^2=-\beta_n^2$, Re $\beta_n>0$. From the solution the potentials are obtained. These are the first non-central potentials for which the Schroedinger equation can be solved in closed form. Case of bound states: these can be "projected out" of $F(k)$, so that with $F(k)=R^{-1}(k)F_N(k)$ and $T(k^2)=[F_N^T(k)F_N(-k)]^{-1}$ the problem is essentially reduced to the case of no bound states. But from the solution of the Gel'fand-Levitan equation one obtains the potentials as well as the bound state wave functions. A number of examples are given. *F. Rohrlich* (Iowa City, Ia.).

Colmez, Jean. Définition de l'opérateur H de Schrödinger pour l'atome d'hydrogène. Ann. Sci. Ecole Norm. Sup. (3) **72** (1955), 111–149.

A relatively elementary rigorous treatment of the formulation of the Hamiltonian operator $H=\Delta+r^{-1}$, where Δ is the Laplacian and $r=(x^2+y^2+z^2)^{\frac{1}{2}}$, as a self-adjoint operator in the Hilbert space of all square-integrable functions on 3-dimensional euclidean space. The author is apparently unaware of the work of T. Kato [Trans. Amer. Math. Soc. **70** (1951), 195–211; MR **12**, 781] where more general formal operators are similarly formulated and examined, but as a different initial domain is used here the author's results are not automatically included in those of Kato. It is shown in particular that the present operator has a non-trivial continuous spectrum. *I. E. Segal* (Chicago, Ill.).

Kiržnic, D. A. On meson-nucleon interactions. Ž. Eksper. Teoret. Fiz. **27** (1954), 6–18. (Russian)

In addition to the usual pseudoscalar and pseudovector interaction between nucleon and meson, the author introduces an interaction which involves the momentum of the nucleon. For a particle in an attractive external field this term gives rise to a repulsive effect at close range and hence such terms might explain repulsive core effects. *A. J. Coleman* (Toronto, Ont.).

See also: Erdélyi, Kennedy and McGregor, p. 1083; Masse, p. 1138; Rahman, p. 1140; Spruch, p. 1088.

Thermodynamics, Statistical Mechanics

Moreau, J. J. Justification statistique de la loi de la diffusion. Actes du colloque sur la diffusion, Montpellier, 1955, pp. 9–15. Publ. Sci. Tech. Ministère de l'Air, Notes Tech. no. **59**, Paris, 1956.

Si l'on définit le coefficient de diffusion D à partir de l'équation

$$(1)\qquad \frac{1}{D}\frac{\partial C}{\partial t}=\frac{\partial^2 C}{\partial x^2}+\frac{\partial^2 C}{\partial y^2}+\frac{\partial^2 C}{\partial z^2},$$

où C est une concentration, il est utile d'examiner comment s'établit cette équation, pour en préciser la validité. La méthode stochastique introduit une probabilité de présence ayant pour densité $p(A,t)$, et une probabilité de passage $f(A,B,h)$ telle que $fd\tau_B$ soit la probabilité pour qu'une molécule située en A à l'instant t se trouve dans le volume $d\tau_B$ autour de B à l'instant $t+h$. f est normale et isotrope, ne dépend que de la distance AB, et son écart type est de la forme λh. L'équation

$$p(B,t+h)=\iiint p(A,t)f(AB,h)d\tau_A$$

entraîne (1), avec $p=C$, $\lambda=2D$.

Les hypothèses d'homogénéité et d'isotropie sont certainement valables dans certains cas simples (diffusion homogène-fortes dilutions) et sont affectées par les parois. *J. Bass* (Paris).

Królikowski, W.; and Rzewuski, J. On the equation for a distinguished component of the state vector. Bull. Acad. Polon. Sci. Cl. III. **4** (1956), 19–28.

The problem of constructing two body equivalent potentials, starting from complicated many body Hamiltonians appears in many fields of modern theoretical physics: e.g. when one has to obtain a nucleon-nucleon

potential from a meson theory or one wants to get an optical potential well starting from the knowledge of two body nuclear forces.

The authors give a rather elegant formal solution of the problem, by using a canonical transformation which enables them to obtain a compact expression for the potential. As the result of a perturbation expansion one gets a second order potential which is closely similar to the conventional one. This potential has an imaginary part which takes into account the anelastic processes.

It is the reviewer's opinion that it would be quite interesting to apply the mathematical method to physically important problems (as meson theory of nuclear forces) and to understand better the relation with other ways of constructing potentials [see, e.g., Brueckner and Watson, Phys. Rev. (2) **90** (1953), 699–708; MR **15**, 379; Henley and Ruderman, ibid. **92** (1953), 1036–1044].

S. Fubini (Chicago, Ill.).

Ishiguro, Eiichi; Arai, Tadashi; Sakamoto, Michiko; and Takayanagi, Kazuo. Tables useful for the calculation of the molecular integrals. VIII. Nat. Sci. Rep. Ochanomizu Univ. **6** (1955), 157–181.

The present part [for part VII see Ishiguro, Yuasa, Sakamoto, Kodaira, and Higuchi, same Rep. **5** (1955), 197–212; MR **17**, 542] gives 23 pp. of formulas for expressing Coulomb integrals in terms of incomplete gamma functions, and ionic integrals in terms of incomplete gamma functions (if two of the orbitals are of s-type) or more involved auxiliary functions (in the general case).

A. Erdélyi (Pasadena, Calif.).

Marmasse, Claude. Sur l'axiomatique d'une nouvelle méthode en mécanique statistique. C. R. Acad. Sci. Paris **242** (1956), 2810–2812.

★**Becker, Richard. Theorie der Wärme.** Springer-Verlag, Berlin-Göttingen-Heidelberg, 1955. viii+320 pp. DM 39.60.

Textbook of thermodynamics and statistical mechanics on the graduate student level. Chapter I (60 pages) contains an exposition of the main features of classical thermodynamics, with illustrative applications (van-der-Waals gas, superconductivity, small droplets). Chapter II first treats the elementary kinetic theory, including Boltzmann's H-theorem. After an introduction into the mechanics of phase space, the microcanonical ensemble is introduced, and the problems in connection with the ergodic hypothesis, irreversibility, and recurrence are discussed. Subsequently, the canonical ensemble is introduced by singling out a small part of a large system; similarly a third ensemble is mentioned, which represents a gas with fixed pressure and fluctuating volume.

In chapter III quantum statistics is treated in the usual way, by postulating the existence of transition probabilities between stationary states of an unperturbed Hamiltonian and computing them by means of first-order perturbation theory. The results are applied in chapter IV to the ideal gas with Fermi and Bose statistics. In addition the cluster expansion for real gases is given and the problem of condensation is discussed. Chapter V is devoted to crystals: specific heat, lattice vibrations, order-disorder, ferromagnetism. In chapter VI the connection between entropy and probability is applied to the calculation of fluctuation phenomena in stationary ensembles, but this part remains somewhat sketchy; more attention is paid to time-dependent fluctuations: Brownian motion, Nyquist relation, etc. Finally, chapter VII deals with irreversible processes, and gives the derivation and various applications of the Onsager relations.

Apart from minor exceptions, the book is written in the same careful and lucid way that made Becker's "Theorie der Elektrizität" famous. The emphasis lies on the main methods and problems, rather than on special applications. The fundamental difficulties of statistical mechanics are discussed in a qualitative, intuitive way, which is appropriate for a first course. [Yet, in the reviewer's opinion, it should have been more emphasized that not all difficulties are overcome; in particular, he does not agree with the statement that quantum-mechanics furnishes an irreproachable proof of the H-theorem].

N. G. van Kampen (Utrecht).

Chang-Chung-Suei. A remark on H theorem. Acta Sci. Sinica **3** (1954), 1–13.

This paper analyzes the relation between two familiar formulations and derivations of the quantum-mechanical H-theorem, one treating the system studied as one single entity and therefore using the occupation probabilites for the quantum states of the whole system, and one considering the system as an assembly of particles and studying the probability distribution of the latter among their available states. It is shown under which conditions the two approaches are equivalent. The author does not treat the system as isolated but assumes for reasons of convenience that it interacts with a reservoir in thermal equilibrium. He further allows to a certain extent for the presence of an interaction between the particles composing the system.

L. Van Hove (Utrecht).

Kac, M. Some remarks on the use of probability in classical statistical mechanics. Acad. Roy. Belg. Bull. Cl. Sci. (5) **42** (1956), 356–361.

Es wird an folgendem Modell [ähnlich dem von P. und T. Ehrenfest, Encyk. Math. Wiss., Bd. IV, Art. 32, Teubner, Leipzig, 1912, aber einfacher] die Bedeutung des Boltzmann'schen Stoßzahlansatzes, der Nichtumkehrbarkeit und der Wahrscheinlichkeitsauffassung erläutert: n schwarze und weiße Moleküle durchwandern in fester Richtung n zyklische Plätze, von denen m ausgezeichnet sind, in der Art, daß jedes in der Zeiteinheit zum nächsten geht und dabei seine Farbe wechselt, falls es dabei auf einen ausgezeichneten Platz kommt. Das Verhältnis der weißen und schwarzen Moleküle wird untersucht. Der Vorgang ist determiniert, umkehrbar und sogar periodisch. Durch Mittelung über alle möglichen Kombinationen von m ausgezeichneten Plätzen unter den n gegebenen wird durch direkte Rechnung gezeigt, dass dies Verhältnis bei $n\to\infty$ (m/n konstant) zu jedem Zeitpunkt gegen $\frac{1}{2}$ strebt.

Wegen der Einfachheit des Modells fällt diese Betrachtung nicht mit der Gibbs zugeschriebenen Idee zusammen, über die möglichen Anfangszustände zu mitteln.

D. Morgenstern (Berlin).

Kümmel, Hermann. Quantentheoretische Begründung der Boltzmannschen Stossgleichung. Z. Physik **143** (1955), 219–232.

The author gives a derivation of the Boltzmann transport equation of gas theory on the basis of the hierarchy of equations satisfied by the distribution functions for one, two, etc. particles. The method is very similar to that of H. S. Green [Proc. Phys. Soc. Sect. A. **66** (1953), 325–332; MR **14**, 1048]. Classical and quantum-mechanic-

al gases are considered. The assumptions needed in the derivation are formulated very clearly. *L. Van Hove.*

Tietz, T. Über eine Approximation der Fermischen Verteilungsfunktion. Ann. Physik (6) **15** (1955), 186–188.

The author lists a large number of methods for an approximate analytic treatment of the Thomas-Fermi equation $y''=x^{-1/2}y^{3/2}$. Starting from a remark by Brinkman, he derives the simple approximation $y=(1+x/a)^{-2}$ with $a\approx 1.860$, which deviates only slightly from the numerically well-known exact solution. *P.-O. Löwdin.*

Farinelli, U.; and Gamba, A. Entropy in quantum mechanics. Nuovo Cimento (10) **3** (1956), 1033–1044.

The question of how to justify the diagonalization of the density matrix, which is necessary in quantum statistics to obtain approach to equilibrium, is here answered by an appeal to information theory. On the assumption that the density matrix does not represent the physical state of a system, but only our partial knowledge of it, it is argued that off-diagonal elements must be put zero. Moreover, the weights attached to states that cannot be distinguished experimentally, must be put equal. This is Tolman's argument [The principles of statistical mechanics, Oxford, 1938] applied to non-stationary states; the reviewer disagrees thoroughly with this point of view.
N. G. van Kampen (Utrecht).

Siegel, Armand. Stochastic basis of Onsager's minimum principle. Phys. Rev. (2) **102** (1956), 953–959.

From Langevin's equation $\dot{\alpha}+s\alpha=\varepsilon(t)$, where $\varepsilon(t)$ is a randomly varying force, Onsager and Machlup [Phys. Rev. (2) **91** (1953), 1505–1512; MR **15**, 273] derived a probability distribution for the "paths" $\alpha(t)$ $(0<t<\tau)$ with fixed $\alpha(0)$, namely

$$(*)\qquad \text{const}\times\exp\Big[-(s/k)\int_0^\tau\{\dot{\alpha}(t)+s\alpha(t)\}^2dt\Big].$$

The macroscopic law, viz. $\dot{\alpha}=-s\alpha$, is obtained by maximizing (*). However, it is here shown, that if the distribution of the integral in the exponent of (*) is calculated on the basis of this path distribution, it turns out not to be peaked about the value 0, owing to the fact that the occurrence of zigzag paths is favored by their overwhelming majority. In fact, the mean value tends to infinity, when the number of "gates" that are used to describe the path is increased. The difficulty can be overcome, provided that the number of gates is small (compared to the initial entropy fluctuation $\frac{1}{2}s\alpha(0)^2$ in units k). This restriction does not materially affect the applicability of (*). *N. G. van Kampen* (Utrecht).

Green, H. S. Molecular theory of irreversible processes in fluids. Proc. Phys. Soc. Sect. B. **69** (1956), 269–280.

The aim is to generalize the methods of statistical mechanics of transport processes in rare gases in such a way that they can be applied to dense fluids. The usual hierarchy of distribution functions

$$f_q(\mathbf{x}_1,\cdots,\mathbf{x}_q;\,\xi_1,\cdots,\xi_q;\,t)$$

is replaced with a new, but related, set of probability distributions F_q. The various thermodynamic quantities are written in terms of the F_q, and an expression for the entropy in non-equilibrium states is suggested. A hierarchy of equations, expressing $\partial F_q/\partial t$ in terms of F_q and F_{q+1} is obtained. Use is made of a generalized Stosszahlansatz to the effect that molecules entering the volume to which the F_q refer are uncorrelated with the ones already inside. In this way a generalized Boltzmann equation is obtained. On this basis the H-theorem is proved by showing that the time-derivative of the entropy cannot be negative. Because of the simple relation between the F_q and the theory of the grand partition function, it is hoped that they will furnish an easier access to non-equilibrium fluids than the f_q. *N. G. van Kampen* (Utrecht).

Waldmann, L. Über die Relaxationszeiten und Eigenfunktionen des Maxwellschen Gases. Z. Naturf. **11a** (1956), 523–524.

Temperley, H. N. V. Combinatorial problems suggested by the statistical mechanics of domains and of rubber-like molecules. Phys. Rev. (2) **103** (1956), 1–16.

Several problems are considered dealing primarily with random walks on a plane square lattice subject to various constraints. Two goals, which the author was unable to achieve, were to find the multiple generating function for perimeter and area of single domains and the generating function for the length of paths with no multiple points. The first of these is suggested by some questions dealing with domains in the Ising model of ferromagnetism, whereas the latter is suggested by a model of chain polymers. The author constructs a number of generating functions describing situations which would seem to be quite similar to the above and by studying the properties of these solvable problems tries to infer certain properties of the unsolved problems.

The paper contains a number of worthwhile suggestions but one particularly interesting result is a demonstration that the "attrition coefficient" for the chain polymer and the singular point of the Ising model on corresponding lattices should be simply related. *G. Newell.*

See also: Brot, p. 1157; Dyson, p. 1165.

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